

**Nonlinear response of the trap model in the aging regime: Exact results in the strong-disorder limit**

Cécile Monthus

*Service de Physique Théorique, Unité de Recherche associée au CNRS, DSM/CEA Saclay, 91191 Gif-sur-Yvette, France*

(Received 17 June 2003; published 10 February 2004)

We study the dynamics in the one-dimensional disordered trap model with a broad distribution of trapping times  $p(\tau) \sim 1/\tau^{1+\mu}$ , when an external force is applied from the very beginning at  $t=0$ , or only after a waiting time  $t_w$ , in the linear as well as in the nonlinear response regime. Using a real-space renormalization procedure that becomes exact in the limit of strong disorder  $\mu \rightarrow 0$ , we obtain explicit results for many observables, such as the diffusion front, the mean position, the thermal width, the localization parameters and the two-particle correlation function. In particular, the scaling functions for these observables give access to the complete interpolation between the unbiased case and the directed case. Finally, we discuss in detail the various regimes that exist for the average position in terms of the two times and the external field.

DOI: 10.1103/PhysRevE.69.026103

PACS number(s): 02.50.-r, 75.10.Nr, 05.20.-y

**I. INTRODUCTION**

Trap models provide a simple phenomenological mechanism for aging [1–3]. Their aging properties have thus been much studied, either in the mean-field version [4–7], where “usual aging” occurs, or in the one-dimensional version [8,9], where both aging and subaging behaviors appear in different correlation functions. The mathematicians have also been interested by these trap models [10] with special attention to the cases  $d=1$  [11,12] and  $d=2$  [13]. The one-dimensional version is moreover interesting on its own, since it appears in various physical applications concerning for instance transport properties in disordered chains [14,15] or the dynamics of denaturation bubbles in random DNA sequences [16].

The study of the response to an external field and its relation with the thermal fluctuations has been for some years a central question in the description of the aging dynamics of glassy systems [17,18]. It is thus natural to consider the trap models from this point of view. The studies on the violation of the fluctuation-dissipation relation in mean-field trap models have shown that the results depend on the observable [19], and on the choice of functional form of the hopping rates [20,21]. For trap models on a hypercubic lattice, there are no such ambiguities in the choice of observables and external fields, since the natural observable is the position: one is interested into the response of the position to an external bias, and in the thermal fluctuations of the position. Recently, the response in the one-dimensional trap model was studied via scaling arguments and numerical simulations in Ref. [22], where various regimes were found depending on the relative values of the two times considered ( $t_w, t_w + t$ ) and the external applied field  $f$ , the main results being that in the linear response regime, the fluctuation-dissipation relation (or Einstein’s relation) is still valid in the aging sector, whereas the response always become non-linear at long times. We have shown in Ref. [23] that these two response properties could be understood as consequences of a “non-linear Fluctuation Theorem” that originates from a very special dynamical property of the trap model. In this paper, we consider again the response properties of the trap model, but with a complementary point of view: we use a real-space

renormalization group (RSRG) procedure to derive various explicit exact results in the limit of high disorder.

The RSRG methods, that have appeared in the field of disordered quantum spin chains [24,25], have then been very powerful to study the Sinai model [26], as well as reaction-diffusion processes in a Brownian potential [27]. In particular, since the out-of-equilibrium dynamics of the random field Ising model (RFIM) can be described as a reaction-diffusion process in a Brownian potential for the domain walls, the RSRG method has been used [28] to study the response of the RFIM to an applied external magnetic field. The RSRG approach is also very appropriate to study one-dimensional trap models with a broad distribution of trapping times  $p(\tau) \sim 1/\tau^{1+\mu}$  in the limit of high disorder  $\mu \rightarrow 0$ , as explained in details in Ref. [29] for the *directed* trap model, and in Ref. [9] for the *symmetric* trap model (i.e., in the absence of an external bias): the RSRG method is able to reproduce the exact exponents of the whole aging phase  $0 < \mu < 1$  and moreover allows one to compute exact scaling functions for all observables in a systematic perturbation expansion in  $\mu$  [9,29]. In contrast with other usual methods for disordered systems, the disorder average is not performed at the beginning but at the very end: the RSRG procedure is defined sample by sample, all observables are then evaluated in terms of the relevant properties of a given sample, and can be then averaged with the appropriate measure over the samples. The RSRG approach thus provides a very clear insight into the important dynamical processes.

In this paper, we generalize the RSRG approach described in Ref. [9] for the *unbiased* trap model to include the influence of an external bias, and we obtain exact results for various observables in the high disorder limit  $\mu \rightarrow 0$ . The paper is organized as follows. In Sec. II, we describe the trap model in an external force field  $f$ . In Sec. III, we explain the real-space renormalization procedure in the presence of an external field. In Sec. IV, we describe the effective dynamics when the external field is applied from the initial time  $t = 0$ . In Sec. V, we give explicit results for various one-time observables, such as the diffusion front (78) the mean position (84,85), or the thermal width (91,92). In Sec. VI, we discuss the effective dynamics when the external field is applied only after a waiting time  $t_w$ . In Sec. VII, we compute

the corresponding two-time observables, such as the distribution of the relative displacement (130) and the mean position (133). In Sec. VIII, we summarize our results for the disordered averaged mean position in the various regimes defined by the two times ( $t_w, t$ ) and the external field  $f$ . In Sec. IX, we discuss the rare events that are responsible for the response, in the time sector where the effective dynamics gives no contribution. Finally, in Sec. X, we compare the response and the thermal fluctuations in a given sample, to discuss the validity of the fluctuation-dissipation relation. The conclusions are given in Sec. XI, and the Appendix contains more technical details.

## II. MODELS AND NOTATIONS

### A. Master equation in a sample

To study the trap model in an external force field  $f$ , we consider the following Master Equation [15,23]

$$\frac{dP_t^{(f)}(x)}{dt} = P_t^{(f)}(x+1)W_{\{x+1 \rightarrow x\}}^{(f)} + P_t^{(f)}(x-1)W_{\{x-1 \rightarrow x\}}^{(f)} - P_t^{(f)}(x)[W_{\{x \rightarrow x+1\}}^{(f)} + W_{\{x \rightarrow x-1\}}^{(f)}] \quad (1)$$

with the initial condition  $P_{t=0}^{(f)}(x) = \delta_{x,0}$ . The hopping rates

$$W_{\{x \rightarrow x \pm 1\}}^{(f)} = e^{-\beta E_x \pm \beta f/2} \quad (2)$$

satisfy the detailed balance condition

$$\frac{W_{\{x \rightarrow x+1\}}}{W_{\{x+1 \rightarrow x\}}} = e^{\beta(U_x - U_{x+1})}, \quad (3)$$

where the total energy

$$U_x = -E_x - fx \quad (4)$$

contains both the random energy ( $-E_x$ ) of the trap  $x$  and the potential energy ( $-fx$ ) linear in the position  $x$  induced by the external applied field  $f$ .

### B. Law for the disorder

The trap energies  $\{E_x\}$  are quenched random variables distributed exponentially [2]

$$\rho(E) = \theta(E > 0) \frac{1}{T_g} e^{-E/T_g} \quad (5)$$

This corresponds for the mean trapping time  $\tau = e^{\beta E}$  to the algebraic law

$$q(\tau) = \theta(\tau > 1) \frac{\mu}{\tau^{1+\mu}} \quad (6)$$

with the temperature-dependent exponent  $\mu = T/T_g$ . At low temperatures  $\mu < 1$ , the mean trapping time  $\int d\tau \tau q(\tau)$  is infinite and this directly leads to aging effects.

### C. Link with the “trap model with asymmetry”

To make the link with the “trap model with asymmetry  $h$ ” studied in [22], we note that in the new time  $\tilde{t} = [2 \cosh \beta(f/2)]t$ , the Master equation (1) becomes

$$\frac{dP_{\tilde{t}}^{(f)}(x)}{d\tilde{t}} = \frac{q_-(f)}{\tau_{x+1}} P_{\tilde{t}}^{(f)}(x+1) + \frac{q_+(f)}{\tau_{x-1}} P_{\tilde{t}}^{(f)}(x-1) - \frac{1}{\tau_x} P_{\tilde{t}}^{(f)}(x), \quad (7)$$

where

$$q_{\pm}(f) = \frac{e^{\pm(\beta f/2)}}{e^{+\beta(f/2)} + e^{-\beta(f/2)}} = \frac{1 \pm h(f)}{2} \quad (8)$$

are the probabilities to jump to the right and to the left when escaping a trap, with the normalization  $q_+ + q_- = 1$ . The asymmetry

$$h(f) = q_+ - q_- = \tanh \beta \frac{f}{2} \quad (9)$$

varies between  $h(f=0) = 0$  for the unbiased case and  $h(f \rightarrow \infty) \rightarrow 1$  for the fully directed case.

As soon as  $h > 0$ , the random walk is expected to become asymptotically directed on large scales. In this article, we will be interested into the case where the crossover towards the directed regime happens on large length scales, i.e., the local asymmetry is very small  $h \ll 1$  or equivalently  $\beta f \ll 1$ . In this regime also considered in Ref. [22], the relation between the force and the asymmetry is at lowest order simply linear

$$h(f) = \frac{\beta f}{2} + O((\beta f)^3) \quad (10)$$

and thus the results of the present paper can be straightforwardly compared with Ref. [22].

### D. Entropy and generalized free energy

As in similar models [30–32], the Shannon entropy

$$S(t) = - \sum_x P_t(x) \ln P_t(x) \quad (11)$$

and the energy  $U(t) = \sum_x P_t(x) U_x$  (4) allows us to define a generalized free energy

$$\mathcal{F}(t) = U(t) - TS(t) = \sum_x P_t(x) [U_x + T \ln P_t(x)]. \quad (12)$$

The detailed balance condition implies that it is a nonincreasing function  $d\mathcal{F}(t)/dt \leq 0$ . The equality with zero is possible only if all currents exactly vanish, corresponding to equilibrium. Here, since we consider the infinite line, the equilibrium cannot be reached and the free energy will decrease with no bounds.

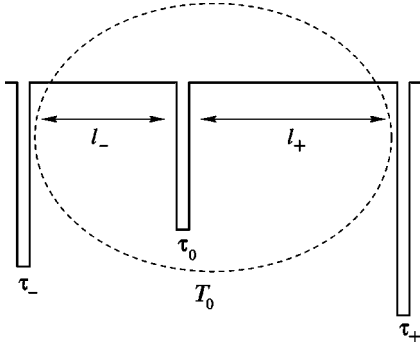


FIG. 1. Definition of the escape time from a trap in the renormalized landscape: the trap of escape time  $\tau_0$  existing in the renormalized landscape at scale  $R$  is surrounded by two renormalized traps that are at distances  $l_+$  and  $l_-$  on each side. The escape time  $T_0$  is the mean time needed to reach either  $\tau_+$  or  $\tau_-$  when starting at  $\tau_0$ .

### III. REAL-SPACE RENORMALIZATION PROCEDURE IN THE PRESENCE OF A FIELD

We have already presented the real-space renormalization procedure for the *unbiased* trap model in Ref. [9]. Here we wish to generalize this approach to the presence of an external bias  $f > 0$ .

#### A. Notion of renormalized landscape at a scale $R$

The basic idea of the real-space renormalization procedure [9,26,29] is that the dynamics at large time is dominated by the statistical properties of the large trapping times. The renormalized landscape at scale  $R$  is defined as follows: all traps with trapping time  $\tau_x < R$  are decimated and replaced by a “flat landscape,” whereas all traps with waiting time  $\tau_x > R$  remain unchanged. At large scale  $R$ , the distribution of the distance  $l$  between two traps of the renormalized landscape at scale  $R$  takes the scaling form

$$P_R(l) \simeq \frac{1}{R^\mu} \mathcal{P}\left(\lambda = \frac{l}{R^\mu}\right), \quad (13)$$

where the scaling distribution is simply exponential  $\mathcal{P}(\lambda) = e^{-\lambda}$

The distribution of the trapping times of the traps in the renormalized landscape at scale  $R$  is simply

$$q_R(\tau) = \theta(\tau > R) \frac{\mu}{\tau} \left(\frac{R}{\tau}\right)^\mu. \quad (14)$$

To relate the renormalization scale  $R$  to the time  $t$ , we have to study the time needed to escape from a renormalized trap.

#### B. “Escape time” from a renormalized trap to another renormalized trap

We now study the escape time  $T_0$  from a trap  $\tau_0$  existing in the renormalized landscape at scale  $R$  in the presence of a field  $f > 0$ . This trap is surrounded by two renormalized traps that are at distances  $l_+$  and  $l_-$  on each side (see Fig. 1).

Whenever the particle escapes from the trap  $\tau_0$ , it escapes to the right with probability  $q_+ = [1 + h(f)]/2$  and to the left with probability  $q_- = [1 - h(f)]/2$  (8).

If it escapes on the right, it will succeed to reach the trap  $\tau_+$  with probability (A10)

$$p_e^+(f, l_+) = \frac{1 - e^{-\beta f}}{1 - e^{-\beta f l_+}}, \quad (15)$$

which varies between  $p_e^+(l, f \rightarrow 0) = 1/l_+$  for the unbiased case and  $p_e^+(l, f \rightarrow \infty) = 1$  for the directed case.

If it escapes on the left, it will succeed to reach the trap  $\tau_-$  with probability (A13)

$$p_e^-(f, l_-) = \frac{(1 - e^{-\beta f})e^{-\beta f(l_- - 1)}}{1 - e^{-\beta f l_-}}, \quad (16)$$

which varies between  $p_e^-(l_-, f \rightarrow 0) = 1/l_-$  for the unbiased case and  $p_e^-(l_-, f \rightarrow \infty) = 0$  for the directed case.

Otherwise, it will be reabsorbed again by the trap  $\tau_0$ . So the total probability to escape when exiting from  $\tau_0$  reads

$$p_e(f, l_+, l_-) = \frac{e^{+\beta(f/2)} p_e^+(l_+, f) + e^{-\beta(f/2)} p_e^-(l_-, f)}{e^{+\beta(f/2)} + e^{-\beta(f/2)}}. \quad (17)$$

#### 1. Number of sojourns in a renormalized trap before escape to a neighboring renormalized trap

As a consequence, the probability  $E_R(n)$  to need  $(1+n)$  successive sojourns in the trap  $\tau_0$  before the particle succeeds to escape either to the trap  $\tau_-$  or to the trap  $\tau_+$  reads

$$E_R(n) = [1 - p_e(f, l_+, l_-)]^n p_e(f, l_+, l_-). \quad (18)$$

For large  $R$ , since we have  $l_\pm = R^\mu \lambda_\pm$  (13) we obtain using Eqs. (A11) and (A14) the scaling form for the probability of escape (17)

$$p_e(f, l_+, l_-) \simeq \frac{1}{R^\mu} \phi\left(F \equiv \beta f R^\mu; \lambda^+ = \frac{l_+}{R^\mu}, \lambda^- = \frac{l_-}{R^\mu}\right) \quad (19)$$

with the scaling function

$$\phi(F; \lambda^+, \lambda^-) = \frac{F(1 - e^{-F(\lambda^+ + \lambda^-)})}{2(1 - e^{-F\lambda^+})(1 - e^{-F\lambda^-})} \quad (20)$$

The expansion near  $F \rightarrow 0$  gives the first correction with respect to the symmetric case studied in Ref. [9]

$$\begin{aligned} \phi(F; \lambda^+, \lambda^-) &= \frac{1}{2} \left[ \frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right] + \frac{\lambda^+ + \lambda^-}{24} F^2 \\ &\quad - \frac{(\lambda^+)^3 + (\lambda^-)^3}{1440} F^4 + O(F^6), \end{aligned} \quad (21)$$

whereas in the other limit  $F \rightarrow \infty$ , we have

$$\phi(F; \lambda^+, \lambda^-) \underset{F \rightarrow \infty}{\simeq} \frac{F}{2} (1 + e^{-F\lambda^+} + e^{-F\lambda^-} + \dots). \quad (22)$$

As a consequence, at large scale  $R$ , the number  $n$  of returns is distributed exponentially

$$E_R(n) \simeq \frac{1}{\langle n \rangle(R, F)} e^{-[n/\langle n \rangle(R, F)]} \quad (23)$$

where the mean number of returns

$$\langle n \rangle(R, F) = \frac{R^\mu}{\phi(F; \lambda^+, \lambda^-)} \quad (24)$$

varies between

$$\langle n \rangle(R, F \rightarrow 0) = R^\mu \frac{2\lambda^+ \lambda^-}{\lambda^+ + \lambda^-} \quad (25)$$

and

$$\langle n \rangle(R, F \rightarrow \infty) \simeq R^\mu \frac{2}{F} = \frac{2}{\beta f}. \quad (26)$$

### 2. Total time spent inside a renormalized trap before escape to a neighboring renormalized trap

Let us now consider the probability distribution  $P_{in}(t_{in})$  of the total time  $t_{in}$  spent inside the trap  $\tau_0$  before its escape. It can be decomposed into the number  $n$  of sojourns, where  $n$  is distributed with (18)

$$t_{in} = \sum_{n=1}^{1+n} t_n, \quad (27)$$

where  $t_n$  is the time spent during the sojourn  $i$  in the trap  $\tau_0$ , so it is distributed with the exponential distribution with mean time  $\tau_0$ . Actually, since  $n$  is large in the large  $R$  limit, we have the central-limit theorem

$$t_{in} \underset{n \rightarrow \infty}{\simeq} n \langle t_n \rangle = n \tau_0. \quad (28)$$

Since the number  $n$  is distributed with exponentially (23), we finally obtain that  $t_{in}$  is also exponentially distributed

$$\hat{P}_{in}(t_{in}) \underset{R \rightarrow \infty}{\simeq} \frac{1}{T_0} e^{-t_{in}/T_0} \quad (29)$$

with the characteristic time

$$T_0 = \tau_0 \langle n \rangle(R, F). \quad (30)$$

Since the smallest trapping times existing in the renormalized landscape at scale  $R$  is  $\tau_0 = R$ , the time spent inside the trap  $\tau_0$  before it succeeds to escape scales as

$$t_{in}(R, F) \underset{R \rightarrow \infty}{\sim} R \langle n \rangle(R, F) = \frac{R^{1+\mu}}{\phi(F; \lambda^+, \lambda^-)}. \quad (31)$$

### 3. Total time spent during the unsuccessful excursions before the escape

Among the  $n$  unsuccessful excursions, there are  $m$  excursions on the left and  $(n-m)$  excursions on the right, where  $m$  is distributed with the binomial distribution  $2^{-n} C_n^m$ . Since  $n$  and  $m$  are large, we again have a central-limit theorem

$$t_{out} = \sum_{n=1}^m t_n^- + \sum_{j=1}^{n-m} t_j^+ \simeq m \theta_{us}^-(f, l_-) + (n-m) \theta_{us}^+(f, l_+), \quad (32)$$

where  $\theta_{us}^\pm(f, l)$  represents the mean time needed to return to 0 when starting at  $(\pm 1)$  without touching the point  $(\pm l)$  in a flat landscape. Using the asymptotic behavior [(A17) and (A21)]

$$\theta_{us}^+(f, l) \underset{l \rightarrow \infty}{\simeq} l \Theta(u = \beta f l) \quad (33)$$

with the scaling function  $\Theta$  given in Eq. (A18), the ratio between  $t_{out}$  and  $t_{in}$  (28) scales as

$$\begin{aligned} \frac{t_{out}}{t_{in}}(R, F) &\simeq \frac{\theta_{us}^-(f, l_-) + \theta_{us}^+(f, l_+)}{2\tau_0} \\ &\sim R^{\mu-1} [\Theta(F\lambda^+) + \Theta(F\lambda^-)], \end{aligned} \quad (34)$$

which varies between (A19)

$$\frac{t_{out}}{t_{in}}(R, F \rightarrow 0) \sim R^{\mu-1} \quad (35)$$

and

$$\frac{t_{out}}{t_{in}}(R, F \rightarrow \infty) \sim R^{\mu-1} \frac{1}{F} = \frac{1}{R\beta f}. \quad (36)$$

We may thus neglect  $t_{out}$  with respect to  $t_{in}$  at large scale  $R$ .

### 4. Time spent during the successful excursion to escape

We finally consider the diffusion time  $t_{diff}^\pm$  of the successful escape to the neighboring renormalized trap ( $l_+$ ) when starting at  $x=1$  without visiting  $x=0$ . The mean time needed to reach  $x=l_+$  when starting at  $x=1$  for a random walk conditioned not to visit  $x=0$  takes the scaling form (A22)

$$t_{diff}^+(f, l_+) \simeq l_+^2 D(u = \beta f l_+), \quad (37)$$

where the scaling function  $D(u)$  is given in Eq. (A23). Using also Eq. (A26), we obtain the scaling

$$\frac{t_{diff}}{t_{in}}(R, F) \sim R^{\mu-1} \mathcal{D}(F\lambda), \quad (38)$$

which is the same as Eq. (34) and thus  $t_{diff}$  is again negligible with respect to  $t_{in}$  (31).

### 5. Conclusion

So we obtain that the total time

$$t_{esc} = t_{in} + t_{out} + t_{diff} \quad (39)$$

needed to escape is actually simply given by the time  $t_{in}$  spent inside the trap  $\tau_0$ . So the distribution of  $t_{esc}$  is given by the exponential (29) with the escape time  $T_0$ .

In conclusion, a trap of the renormalized landscape at scale  $R$  has a trapping time  $\tau$  distributed with Eq. (14), but has an escape time proportional to  $\tau$

$$T_e = \tau \langle n \rangle (R, F) = \tau \frac{R^\mu}{\phi(F; \lambda^+, \lambda^-)} \quad (40)$$

in terms of the two rescaled distances  $\lambda_\pm$  to the neighbors.

#### C. Distribution of escape times in the renormalized landscape

The distribution of the escape time  $T_e$  in the renormalized landscape at scale  $R$  is obtained by averaging over  $\tau, \lambda_\pm$ :

$$\begin{aligned} \mathcal{Q}_R(T_e; f) &= \int_R^{+\infty} d\tau \frac{\mu}{\tau} \left(\frac{R}{\tau}\right)^\mu \int_0^{+\infty} d\lambda^+ d\lambda^- e^{-\lambda^+ - \lambda^-} \delta \\ &\times \left( T_e - \frac{R^\mu}{\phi(F; \lambda^+, \lambda^-)} \tau \right). \end{aligned} \quad (41)$$

It thus takes the scaling form

$$\mathcal{Q}_R(T_e; f) = \frac{1}{R^{1+\mu}} \mathcal{Q}_\mu \left( \tilde{T} = \frac{T_e}{R^{1+\mu}}; F \right), \quad (42)$$

where the scaling function

$$\begin{aligned} \mathcal{Q}_\mu(\tilde{T}; F) &= \frac{\mu}{(\tilde{T})^{1+\mu}} \int_0^{+\infty} d\lambda^+ d\lambda^- e^{-\lambda^+ - \lambda^-} \\ &\times [\phi(F; \lambda^+, \lambda^-)]^{-\mu} \theta \\ &\times \left[ \phi(F; \lambda^+, \lambda^-) > \frac{1}{\tilde{T}} \right] \end{aligned} \quad (43)$$

presents the asymptotic behavior

$$\mathcal{Q}_\mu(\tilde{T}; F) \simeq \frac{\mu}{(\tilde{T})^{1+\mu}} c_\mu(F) \quad (44)$$

with the prefactor

$$\begin{aligned} c_\mu(F) &= \int_0^{+\infty} d\lambda^+ d\lambda^- e^{-\lambda^+ - \lambda^-} [\phi(F; \lambda^+, \lambda^-)]^{-\mu} \\ &= 1 - \mu \int_0^{+\infty} d\lambda^+ d\lambda^- e^{-\lambda^+ - \lambda^-} \ln[\phi(F; \lambda^+, \lambda^-)] \\ &+ O(\mu^2). \end{aligned} \quad (45)$$

For small  $\mu$ , the probability distribution  $\mathcal{Q}_\mu(\tilde{T}, F)$  is dominated by its long tail, and we may approximate it by

$$\mathcal{Q}_\mu(\tilde{T}) \simeq \theta(\tilde{T} > \tilde{T}_\mu(F)) \frac{\mu}{\tilde{T}} \left( \frac{\tilde{T}_\mu(F)}{\tilde{T}} \right)^\mu. \quad (46)$$

The cutoff  $\tilde{T}_\mu(F)$  chosen to preserve the normalization is determined by the coefficient in the long tail part (44) and (45)

$$\tilde{T}_\mu(F) = [c_\mu(F)]^{1/\mu} = e^{S(F)} + O(\mu), \quad (47)$$

where the function  $S(F)$  obtained from Eq. (45) can be computed with the explicit expression (20)

$$S(F) \equiv - \int_0^{+\infty} d\lambda^+ d\lambda^- e^{-\lambda^+ - \lambda^-} \ln[\phi(F; \lambda^+, \lambda^-)] \quad (48)$$

$$= -\ln F + \ln 2 - \gamma_E - \psi\left(1 + \frac{1}{F}\right) - \frac{1}{F} \psi'\left(1 + \frac{1}{F}\right) \quad (49)$$

in terms of  $\psi(z) = \Gamma'(z)/\Gamma(z)$ , and the Euler constant  $\gamma_E = -\psi'(1)$ .

For the unrescaled probability distribution (42), this corresponds to the cutoff

$$T_0(R, f) = R^{1+\mu} e^{S(F)} \quad (50)$$

#### D. Choice of the renormalization scale $R$ as a function of time

The renormalization scale  $R$  has to be chosen as a function of time by the requirement that this effective cutoff  $T_0(R, f)$  is exactly  $t$ , meaning that at time  $t$ , only traps with escape times  $T_e > t$  have been kept, whereas all traps with escape times  $T_e < t$  have been removed and replaced by a flat landscape. So given a renormalization scale  $R$  and an external field  $f$ , the corresponding time  $t$  of the dynamics reads

$$t = t(R, f) \equiv R^{1+\mu} e^{S(F = \beta f R^\mu)}. \quad (51)$$

The renormalization scale  $R(t, f)$  as a function of time  $t$  and field  $f$  is thus defined by the implicit equation

$$t = R^{1+\mu}(t, f) e^{S(\beta f R^\mu(t, f))}. \quad (52)$$

At short times, assuming  $F = \beta f R^\mu(t) \ll 1$ , we may use the expansion

$$S(F) = \ln 2 - 1 - \gamma_E - \frac{1}{12} F^2 + O(F^4) \quad (53)$$

to solve Eq. (52), to obtain the first correction with respect to the unbiased case

$$R(t, f) = \left(\frac{t}{T_0}\right)^{1/(1+\mu)} \left[ 1 + \frac{(\beta f)^2}{12(1+\mu)} \left(\frac{t}{T_0}\right)^{2\mu/(1+\mu)} \dots \right], \quad (54)$$

where  $\tilde{T}_0 = e^{S(0)} = 2e^{-1-\gamma_E}$ . This solution is thus valid for

$$t \ll t_\mu(f) \equiv \left( \frac{1}{\beta f} \right)^{(1+\mu)/\mu}, \quad (55)$$

where the time scale  $t_\mu(f)$  was already shown to play an important role via other approaches [22,23].

At long times, assuming  $F = \beta f R^\mu(t) \gg 1$ , we may use the asymptotic behavior

$$S(F) = -\ln \frac{F}{2} - \frac{\pi^2}{3F} + O\left(\frac{1}{F^2}\right) \quad (56)$$

to solve Eq. (52) which leads to

$$R(t, f) = \frac{\beta f}{2} t \left[ 1 + \frac{\pi^2 2^\mu}{3(\beta f)^{1+\mu} t^\mu} + \dots \right], \quad (57)$$

which is valid for  $t \gg t_\mu(f)$  (55).

As a consequence, the characteristic length scale corresponding to the mean distance between renormalized traps at scale  $R(t)$  (13)

$$\xi(t, f) = R^\mu(t, f) \quad (58)$$

behaves as in the unbiased case at short times [33,34]

$$\xi(t, f) \underset{t \ll t_\mu(f)}{\simeq} t^{\mu/(1+\mu)} [1 + O(\mu)] \quad (59)$$

and as

$$\xi(t, f) \underset{t \gg t_\mu(f)}{\simeq} (\beta f t)^\mu [1 + O(\mu)] \quad (60)$$

at long times.

#### IV. EFFECTIVE DYNAMICS IN THE LIMIT $\mu \rightarrow 0$

##### A. Probabilities to escape on the right or on the left in the renormalized landscape

We are now interested into the relative probability to escape on the right rather than to the left which reads [(15) and (17)] using Eqs. (A11) and (A14)

$$w_+(f, l_+, l_-) \equiv \frac{e^{+\beta(f/2)} p_e^+(f, l_+)}{p_e(f, l_+, l_-)} \simeq W_+(F, \lambda^+, \lambda^-) \quad (61)$$

with the scaling function

$$W_+(F, \lambda^+, \lambda^-) = \frac{1 - e^{-F\lambda^-}}{1 - e^{-F(\lambda^+ + \lambda^-)}}. \quad (62)$$

The complementary probability to escape on the left reads

$$W_-(F, \lambda^+, \lambda^-) = \frac{e^{-F\lambda^-} (1 - e^{-F\lambda^+})}{1 - e^{-F(\lambda^+ + \lambda^-)}}. \quad (63)$$

The expansion near  $F \rightarrow 0$  gives the first correction with respect to the symmetric case [9]

$$W_+(F, \lambda^+, \lambda^-) = \frac{\lambda^-}{\lambda^- + \lambda^+} + \left[ \frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right] \frac{F}{2} + \frac{\lambda^+ - \lambda^-}{12} \left[ \frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right] F^2 + O(F^3), \quad (64)$$

whereas for large  $F$ , the first corrections with respect to the directed case are given by

$$W_+(F, \lambda^+, \lambda^-) \underset{F \rightarrow \infty}{\simeq} 1 - e^{-F\lambda^-} + e^{-F(\lambda^+ + \lambda^-)} + \dots \quad (65)$$

##### B. Rules for the effective dynamics

We thus define the effective dynamics by the following rules:

The particle starting at the origin  $O$  will be at time  $t$  either in the first trap  $M_+$  of the renormalized landscape at scale  $R(t, f)$  on its right or in the first trap  $M_-$  of the renormalized landscape on its left. The weights of the traps  $M_+$  and  $M_-$  are given in terms of Eqs. (62) and (63) by

$$p_{[M_- M_+]}(M_+ | 0) = W_+(F, \lambda^+, \lambda^-), \quad (66)$$

$$p_{[M_- M_+]}(M_- | 0) = W_-(F, \lambda^+, \lambda^-). \quad (67)$$

We now verify that this effective dynamics presents some important properties.

##### C. Consistency upon iteration

The rule for the effective dynamics is consistent upon iteration. Indeed, suppose there are three consecutive traps: the trap  $M_-$  is at a distance  $l_-$  from the origin on the left, the trap  $M_+$  is at a distance  $l_+$  from the origin on the right, and the trap  $M_{++}$  is at a distance  $l$  from the trap  $M_+$  on the right.

Suppose that the trap  $M_+$  is decimated before the traps  $M_-$  and  $M_{++}$ . The new weights for the traps  $M_-$  and  $M_{++}$  become

$$\begin{aligned} p'_{M_-} &= p_{[M_- M_+]}(M_- | 0) \\ &\quad + p_{[M_- M_{++}]}(M_- | M_+) p_{[M_- M_+]}(M_+ | 0) \\ &= p_{[M_- M_{++}]}(M_- | 0) \end{aligned} \quad (68)$$

$$\begin{aligned} p'_{M_{++}} &= p_{[M_- M_{++}]}(M_{++} | M_+) p_{[M_- M_+]}(M_+ | 0) \\ &= \frac{1 - e^{-Fl_-}}{1 - e^{-F(l_+ + l_- + l)}} = p_{[M_- M_{++}]}(M_{++} | 0) \end{aligned} \quad (69)$$

and thus the rules (67) for the occupancies of renormalized traps are consistent upon decimation of traps in the renormalized landscape.

#### D. Conservation of the mean position in any sample for $f=0$

As already emphasized in Refs. [22,23], the trap model has a very special property: when there is no external field  $f=0$  the mean position is a conserved quantity in any given sample, as a simple consequence of the master equation (1) for any realization of the trapping times  $\{\tau_n\}$ . Here in the effective dynamics, the mean position vanishes indeed in each sample with the weights in zero field  $f=0$  given by Eq. (64).

#### E. Nonlinear fluctuation theorem

The nonlinear fluctuation theorem obtained in Ref. [23] says that, in any given sample of the trap model the diffusion front  $P_t^{(+f)}(n)$  in the presence of the external field ( $+f$ ) and the diffusion front  $P_t^{(-f)}(n)$  in the presence of ( $-f$ ) are related by the extremely simple property

$$\frac{P_t^{(+f)}(n)}{P_t^{(-f)}(n)} = e^{\beta f n}. \quad (70)$$

In the effective dynamics, this property is satisfied since the renormalization landscape at time  $t$  is the same for ( $+f$ ) and ( $-f$ ), so the two possible positions  $M_+$  and  $M_-$  are the same, and the weights [(62) and (63)] satisfy the properties

$$W_+(-F, \lambda^+, \lambda^-) = e^{-F\lambda^+} W_+(F, \lambda^+, \lambda^-), \quad (71)$$

$$W_-(-F, \lambda^+, \lambda^-) = e^{F\lambda^-} W_-(F, \lambda^+, \lambda^-). \quad (72)$$

### V. OBSERVABLES WHEN THE FIELD IS APPLIED FROM THE BEGINNING $t \geq 0$

#### A. Diffusion front in a given sample

In the effective dynamics, the diffusion front in a given sample takes the scaling form

$$P_t^{(0)}(n; f) = \frac{1}{\xi(t, f)} \mathcal{P}^{(0)} \left( X = \frac{n}{\xi(t, f)}; F = \beta f \xi(t, f) \right), \quad (73)$$

where the characteristic length scale  $\xi(t, f)$  has been defined in Eqs. (52) and (58) from the renormalization landscape, and where the scaling function reads according to the effective dynamics (67)

$$\begin{aligned} \mathcal{P}^{(0)}(X, F) &= \frac{e^{-F\lambda^-} (1 - e^{-F\lambda^+})}{1 - e^{-F(\lambda^+ + \lambda^-)}} \delta(X + \lambda^-) \\ &+ \frac{1 - e^{-F\lambda^-}}{1 - e^{-F(\lambda^+ + \lambda^-)}} \delta(X - \lambda^+) \end{aligned} \quad (74)$$

in terms of the two rescaled distances  $\lambda_{\pm}$  between the origin and the nearest renormalized traps. The joint distribution of the two rescaled distances is completely factorized

$$D(\lambda^+, \lambda^-) = \theta(\lambda^+) \theta(\lambda^-) e^{-\lambda^+ - \lambda^-}. \quad (75)$$

We now compute the various observables that can be obtained from the sample-dependent diffusion front (74) and the measure (75) over the samples.

#### B. Disorder averaged diffusion front

The averaged diffusion front takes the scaling form

$$\overline{P_t(n, f)} = \frac{1}{\xi(t, f)} g_{\mu} \left( X = \frac{n}{\xi(t, f)}, F = \beta f \xi(t, f) \right), \quad (76)$$

where the scaling function  $g_{\mu}$  reads at lowest order in  $\mu \rightarrow 0$

$$\begin{aligned} g_0(X, F) &= e^{-|X|} [\theta(X > 0) + \theta(X < 0) e^{-F|X|}] \\ &\times \int_0^{+\infty} d\lambda e^{-\lambda} \frac{1 - e^{-F\lambda}}{1 - e^{-F(|X| + \lambda)}} \end{aligned} \quad (77)$$

$$\begin{aligned} &= e^{-|X|} [\theta(X > 0) + \theta(X < 0) e^{-F|X|}] \\ &\times \sum_{n=0}^{+\infty} \frac{F}{(1 + Fn)(1 + F + Fn)} e^{-nF|X|}, \end{aligned} \quad (78)$$

which interpolates between the diffusion fronts for the unbiased case [9] as  $F \rightarrow 0$  and for the directed case [29] as  $F \rightarrow \infty$ . Here, as for other results below, we have given two equivalent forms for the diffusion front: the first one with the integral is more appropriate to study the small  $F$  behavior, whereas the second one with the infinite series is more suited for large  $F$ .

The simple property

$$g_0(-X, F) = e^{-FX} g_0(X, F) \quad (79)$$

is actually expected to be valid for arbitrary  $\mu$  as a consequence of a nonlinear fluctuation theorem, as discussed in Ref. [23].

#### C. Disorder averaged mean position

The disorder average of the mean position takes the scaling form

$$\overline{\langle x(t, f) \rangle} \equiv \sum_{x=-\infty}^{+\infty} x P_t(x, f) = \xi(t, f) \mathcal{X}_{\mu}(F = \beta f \xi(t, f)), \quad (80)$$

where the scaling function  $\mathcal{X}_{\mu}$  reads at lowest order in  $\mu \rightarrow 0$

$$\begin{aligned} \mathcal{X}_0(F) &\equiv \int_{-\infty}^{+\infty} dX X g_0(X, F) \\ &= \int_0^{+\infty} d\lambda \lambda e^{-\lambda} \frac{\frac{F\lambda}{2} \coth \frac{F\lambda}{2} - 1}{F} \\ &= 1 - \frac{1}{F} - \frac{1}{F^3} \psi'' \left( 1 + \frac{1}{F} \right). \end{aligned} \quad (81)$$

The series expansion for small  $F$  reads

$$\mathcal{X}_0(F) = \frac{F}{2} - \frac{F^3}{6} + \frac{F^5}{6} + O(F^7), \quad (82)$$

whereas the asymptotic behavior for large  $F$  reads

$$\begin{aligned} \mathcal{X}_0(F) &= 1 - \frac{1}{F} + \sum_{n=1}^{+\infty} \frac{2}{(1+Fn)^3} \\ &= 1 - \frac{1}{F} + \frac{2\zeta(3)}{F^3} - \frac{\pi^4}{15F^4} + O\left(\frac{1}{F^5}\right). \end{aligned} \quad (83)$$

Using  $F = \beta f \xi(t, f)$  and Eqs. (59) and (60), we finally obtain the expansions

$$\begin{aligned} \overline{\langle x(t, f) \rangle} &= \frac{\beta f}{2} \left(\frac{t}{T_0}\right)^{\mu/(1+\mu)} \left[ 1 - \frac{1}{3} (\beta f)^2 \left(\frac{t}{T_0}\right)^{2\mu/(1+\mu)} \right. \\ &\quad \left. + O((\beta f t^{\mu/(1+\mu)})^4) \right] \end{aligned} \quad (84)$$

at short times, i.e., for  $t \ll t_\mu(f)$  (55), and

$$\overline{\langle x(t, f) \rangle} = \left(\frac{\beta f}{2}\right)^\mu t^\mu - \frac{1}{\beta f} + O\left(\frac{1}{((\beta f)^2 + \mu t^\mu)}\right) \quad (85)$$

for  $t \gg t_\mu(f)$ . The leading terms are thus in agreement with the scaling analysis presented in Ref. [22].

#### D. Thermal width

The disorder averaged thermal width (74) presents the scaling form

$$\begin{aligned} \overline{\langle \Delta x^2(t, f) \rangle} &\equiv \overline{\sum_{x=-\infty}^{+\infty} x^2 P_t(x, f) - \left[ \sum_{x=-\infty}^{+\infty} x P_t(x, f) \right]^2} \\ &= \xi^2(t, f) \Delta_\mu(F = \beta f \xi(t, f)), \end{aligned} \quad (86)$$

where the scaling function reads at lowest order in  $\mu$

$$\begin{aligned} \Delta_0(F) &= \int_0^{+\infty} d\lambda e^{-\lambda} \frac{\lambda^2}{4H} \left[ \coth \frac{F\lambda}{2} - \frac{\frac{F\lambda}{2}}{\sinh^2 \frac{F\lambda}{2}} \right] \\ &= \frac{1}{F} + \frac{2}{F^4} \psi'' \left( 1 + \frac{1}{F} \right) + \frac{1}{F^5} \psi''' \left( 1 + \frac{1}{F} \right). \end{aligned} \quad (87)$$

It is thus directly related to the scaling function  $\mathcal{X}_0(F)$  governing the mean displacement (81) via the simple relation

$$\Delta_0(F) = \left[ \frac{1}{F} + \frac{d}{dF} \right] \mathcal{X}_0(F). \quad (88)$$

Using the asymptotic behaviors for small and large  $F$

$$\Delta_0(F) = 1 - \frac{2}{3} F^2 + F^4 + O(F^6), \quad (89)$$

$$\Delta_0(F) = \frac{1}{F} - \frac{4\zeta(3)}{F^4} + \frac{\pi^4}{5F^5} + O\left(\frac{1}{F^6}\right) \quad (90)$$

with  $F = \beta f \xi(t, f)$  [(59) and (60)], we obtain the following leading terms for the unrescaled thermal width

$$\begin{aligned} \overline{\langle \Delta x^2(t, f) \rangle} &= \left(\frac{t}{T_0}\right)^{2\mu/(1+\mu)} \left[ 1 - \frac{2}{3} (\beta f)^2 \left(\frac{t}{T_0}\right)^{2\mu/(1+\mu)} \right. \\ &\quad \left. + O((\beta f)^4 t^{4\mu/(1+\mu)}) \right] \end{aligned} \quad (91)$$

for  $t \ll t_\mu(f)$  (55), and

$$\begin{aligned} \overline{\langle \Delta x^2(t, f) \rangle} &= \frac{\left(\frac{\beta f}{2}\right)^\mu t^\mu}{\beta f} \left[ 1 - \frac{4\zeta(3)}{[(\beta f)^{1+\mu} t^\mu]^3} \right. \\ &\quad \left. + O\left(\frac{1}{[(\beta f)^{1+\mu} t^\mu]^4}\right) \right] \end{aligned} \quad (92)$$

for  $t \gg t_\mu(f)$ .

#### E. Mean square displacement

In addition to the thermal width (86), it is also interesting to consider separately the two terms, namely, the mean square displacement

$$\begin{aligned} \mathcal{D}_0(F) &\equiv \overline{\langle X^2 \rangle} \\ &= 2 - \frac{2}{F} + \frac{2}{F^2} + \frac{2}{F^4} \psi'' \left( 1 + \frac{1}{F} \right) \\ &= 1 + \frac{F^2}{3} - \frac{F^4}{3} + O(F^6) \end{aligned} \quad (93)$$

and the square of the mean displacement

$$\overline{\langle X \rangle^2} = 2 - \frac{3}{F} + \frac{2}{F^2} - \frac{1}{F^5} \psi''' \left( 1 + \frac{1}{F} \right) = F^2 - \frac{4F^4}{3} + O(F^6) \quad (94)$$

in the rescaled variable  $X = x/\xi(t, f)$  as usual.

#### F. Localization parameters

The localization parameters, defined as the disorder averages of the probabilities to find  $k$  independent particle in the same trap, follow the scaling form

$$\overline{Y_k^{(\mu)}(t, f)} \equiv \sum_{x=-\infty}^{+\infty} \overline{[P_t^{(f)}(x)]^k} = \mathcal{Y}_k^{(\mu)}(F = \beta f \xi(t, f)), \quad (95)$$

where the scaling function at lowest order in  $\mu$  reads (74)



$$\begin{aligned}
 \mathcal{Y}_k^{(0)}(F) &\equiv \int_0^{+\infty} d\lambda^+ \int_0^{+\infty} d\lambda^- e^{-\lambda^+ - \lambda^-} \\
 &\times \left( \left[ \frac{e^{-F\lambda^-} (1 - e^{-F\lambda^+})}{1 - e^{-F(\lambda^+ + \lambda^-)}} \right]^k + \left[ \frac{1 - e^{-F\lambda^-}}{1 - e^{-F(\lambda^+ + \lambda^-)}} \right]^k \right) \\
 &= \sum_{m=0}^{+\infty} \frac{k\Gamma\left(\frac{1}{F} + m\right)\Gamma(k+m)}{F\Gamma(1+m)\Gamma\left(1 + \frac{1}{F} + k+m\right)} \\
 &\times \left[ \frac{1}{1+Fm} + \frac{1}{1+F(k+m)} \right]. \quad (96)
 \end{aligned}$$

The series expansion for small  $F$  gives the first correction with respect to the unbiased case [9]

$$\begin{aligned}
 \mathcal{Y}_k^{(\mu)}(F) &= \frac{2}{k+1} + F^2 \frac{2k(k-1)}{(k+1)(k+2)(k+3)} \\
 &+ F^4 \frac{2k(k-1)(k^2-15k-4)}{(k+1)(k+2)(k+3)(k+4)(k+5)} \\
 &+ O(F^6), \quad (97)
 \end{aligned}$$

whereas for large  $F$ , the first corrections with the directed case [29] read

$$\begin{aligned}
 \mathcal{Y}_k^{(0)}(F) &= 1 - [\gamma_{Euler} + \psi(k)] \frac{1}{F} \\
 &+ \left( \frac{[\gamma_{Euler} + \psi(k)]^2}{2} + \frac{\pi^2}{4} - \frac{3}{2} \psi'(k) \right) \frac{1}{F^2} \\
 &+ O\left(\frac{1}{F^3}\right). \quad (98)
 \end{aligned}$$

For  $k=2$  and  $k=3$ , we may moreover sum the series (96) to obtain

$$\mathcal{Y}_2^{(0)}(F) = 1 - \frac{1}{F} + \frac{2}{F^2} - \frac{2}{F^3} \psi'\left(1 + \frac{1}{F}\right) \quad (99)$$

$$\mathcal{Y}_3^{(0)}(F) = 1 - \frac{3}{2F} + \frac{3}{F^2} - \frac{3}{F^3} \psi'\left(1 + \frac{1}{F}\right) \quad (100)$$

which are thus related by the very simple relation

$$\frac{1 - \mathcal{Y}_2^{(0)}(F)}{2} = \frac{1 - \mathcal{Y}_3^{(0)}(F)}{3}. \quad (101)$$

This relation actually reflects the two- $\delta$  structure of the diffusion front in any given sample: indeed, if we note  $p_1$  and  $p_2$  the weights of the two- $\delta$  peaks with  $p_1 + p_2 = 1$ , it is immediate to obtain the relation (101) since

$$1 - Y_2^{(0)} = (p_1 + p_2)^2 - p_1^2 - p_2^2 = 2p_1p_2,$$

$$1 - Y_3^{(0)} = (p_1 + p_2)^3 - p_1^3 - p_2^3 = 3p_1p_2(p_1 + p_2) = 3p_1p_2. \quad (102)$$

### G. Entropy

The entropy (11) is closely related to the localization parameters (95) since it reads

$$\begin{aligned}
 \overline{S(t, f)} &\equiv - \sum_{x=-\infty}^{+\infty} \overline{P_t(x) \ln P_t(x)} \\
 &= - \left[ \frac{\partial Y_k^{(\mu)}(t, f)}{\partial k} \right]_{k=1} \\
 &= S^{(\mu)}(F = \beta f \xi(t, f)), \quad (103)
 \end{aligned}$$

where the scaling function reads at lowest order in  $\mu$

$$\begin{aligned}
 S^{(0)}(F) &= \frac{\psi''\left(1 + \frac{1}{F}\right)}{F^2} + \sum_{m=0}^{+\infty} \left( \frac{1}{(1+F+Fm)^2} - \frac{1}{(1+Fm)^2} \right) \\
 &\times \left[ \psi(1+m) - \psi\left(1 + \frac{1}{F} + m\right) \right]. \quad (104)
 \end{aligned}$$

The asymptotic behaviors are given by the series

$$S^{(0)}(F) = \frac{1}{2} - \frac{F^2}{12} + \frac{F^4}{20} + O(F^6) \quad (105)$$

for small  $F$  and by

$$S^{(0)}(F) = \frac{\pi^2}{6F} - \frac{3\zeta(3)}{F^2} + O\left(\frac{1}{F^3}\right) \quad (106)$$

for large  $F$ . So for the unbiased case  $f=0$ , the entropy remains forever frozen at the value  $S^{(0)}(F=0) = 1/2$ . For the biased case, the entropy decays towards zero as the directed character gets stronger as time grows.

### H. Two-particle correlation function

The two-particle correlation function reads

$$\begin{aligned}
 C(l, t) &\equiv \sum_{x=-\infty}^{+\infty} \sum_{x'=-\infty}^{+\infty} \overline{P_t(x) P_t(x')} \delta_{l, |x-x'|} \\
 &\simeq_{t \rightarrow \infty} Y_2^{(0)} \delta_{l,0} + \frac{1}{\xi(t, f)} C_\mu \left( \lambda = \frac{l}{\xi(t, f)} \right), \quad (107)
 \end{aligned}$$

where the weight of the  $\delta$  peak corresponds as it should to the localization parameter  $Y_2^{(0)}$  discussed above, whereas the scaling function of the long-ranged part reads at lowest order

$$C_0(\lambda, F) = e^{-\lambda} \frac{1 - 2F\lambda e^{-F\lambda} - e^{-2F\lambda}}{F(1 - e^{-F\lambda})^2}. \quad (108)$$

In particular, the first correction with respect to the unbiased case reads [9]

$$C_\mu^{(0)}(\lambda, F) = e^{-\lambda} \frac{\lambda}{3} \left[ 1 - \frac{F^2 \lambda^2}{30} + O(F^4 \lambda^4) \right]. \quad (109)$$

## VI. EFFECTIVE DYNAMICS WHEN THE FIELD $f$ IS APPLIED FOR $t \geq t_w$

We now study the dynamics in the following aging ‘‘experiment’’ already considered in Ref. [22]: the system first evolves with no external field  $f=0$  during the interval  $t \in [0, t_w]$ , and then an external field  $f>0$  is applied for  $t \in [t_w, +\infty)$ .

### A. Time sector governed by the effective dynamics

The scale of the renormalized landscape corresponding to time  $t_w$  and to no external field  $f=0$  reads (54)

$$R(t_w, f=0) = \left( \frac{t_w}{T_0} \right)^{1/(1+\mu)}. \quad (110)$$

The corresponding length scale between renormalized traps is given by (59)

$$\xi(t_w, f=0) = \left( \frac{t_w}{T_0} \right)^{\mu/(1+\mu)}. \quad (111)$$

The state reached at  $t_w$ , which is made out of two- $\delta$  peaks (74), has to be considered as an initial condition for the dynamics in the presence of  $f>0$  in the new time ( $t - t_w$ ). Since at  $t_w$ , the particle is typically in a trap of trapping time  $\tau > R(t_w, f=0)$ , the effective dynamics corresponds to no move as long as  $R(t - t_w, f) < R(t_w, 0)$ , whereas the effective dynamics governed by the decimation procedure becomes ‘‘active’’ for  $R(t - t_w, f) > R(t_w, f=0)$ . As a consequence, it is useful to introduce the parameter

$$\alpha(t, t_w, f) \equiv \frac{\xi(t - t_w, f)}{\xi(t_w, f=0)} = \frac{R^\mu(t - t_w, f)}{R^\mu(t_w, f=0)} \quad (112)$$

that measures the ratio of the length scales of renormalized landscape at the two scales  $R(t - t_w, f)$  and  $R(t_w, f=0)$ . The response in the time sector  $\alpha < 1$ , which is governed by rare events will be discussed separately in the Sec. IX. In this section, we will concentrate on the time sector  $\alpha(t, t_w, f) > 1$ , where the effective dynamics governed by the decimation procedure is the leading effect.

### B. Role of the characteristic time $t_\mu(f)$ associated to the external field

We have already seen that a bias  $f$  introduces the characteristic time scale  $t_\mu(f) = (1/\beta f)^{(1+\mu)/\mu}$  (55). It is thus natural that the domain in  $(t - t_w)$  corresponding to the time sector  $\alpha > 1$  actually depends on the relative value of  $t_w$  with respect to  $t_\mu(f)$ .

For  $t_w \ll t_\mu(f)$ , the time sector  $\alpha > 1$  corresponds to the domain  $(t - t_w) > t_w$ , and the ratio  $\alpha$  interpolates between

$$\alpha(t, t_w, f) = \left( \frac{t - t_w}{t_w} \right)^{\mu/(1+\mu)} \quad \text{for } t_w < t - t_w \ll t_\mu(f) \quad (113)$$

and

$$\alpha(t, t_w, f) = \left( \frac{\beta f}{2} (t - t_w) \right)^\mu \quad \text{for } t - t_w \gg t_\mu(f). \quad (114)$$

On the other hand, for  $t_w \gg t_\mu(f)$ , the the time sector  $\alpha > 1$  corresponds to the domain  $(t - t_w) > (2/\beta f) t_w^{1/\mu}$ , and we have Eq. (114) in the whole time sector  $\alpha > 1$ .

To simplify the reading in the following, we will use the simplified notations

$$R \equiv R(t - t_w, f),$$

$$R_w \equiv R(t_w, f=0),$$

$$\alpha \equiv \alpha(t, t_w, f) = \frac{R^\mu}{R_w^\mu}. \quad (115)$$

### C. Statistical properties of the renormalized landscape at two successive scales

To study the two-time effective dynamics, we will need the probability measure for the renormalized landscape at the two successive scales  $R_w$  and  $R$ . Since the two sides of the origin are independent, we first consider the half line  $x > 0$  alone. The joint probability that the first renormalized trap at scale  $R_w$  is at distance  $l_w^+$ , has a trapping time  $\tau_w^+$  and that the first renormalized trap at scale  $R$  is at distance  $l^+$  reads

$$\begin{aligned} & A_{R_w, R}(\tau_w^+, l_w^+; l^+) \\ &= \theta(\tau_w^+ > R_w) \frac{\mu}{(\tau_w^+)^{1+\mu}} e^{-l_w^+/R_w^\mu} \\ & \times \left[ \theta(\tau_w^+ > R) \delta(l^+ - l_w^+) \right. \\ & \left. + \theta(\tau_w^+ < R) \theta(l^+ > l_w^+) \frac{1}{R^\mu} e^{-(l^+ - l_w^+)/R^\mu} \right]. \end{aligned} \quad (116)$$

After integration over the trapping time  $\tau_w^+$ , we obtain the scaling form

$$\begin{aligned} & A_{R_w, R}(l_w^+; l^+) \equiv \int d\tau_w^+ A_{R_w, R}(\tau_w^+, l_w^+; l^+) \\ &= \frac{1}{R_w^\mu R^\mu} \mathcal{A} \left( \lambda_w^+ \equiv \frac{l_w^+}{R_w^\mu}; \lambda^+ \equiv \frac{l^+ - l_w^+}{R^\mu}; \alpha \equiv \frac{R^\mu}{R_w^\mu} \right) \end{aligned} \quad (117)$$

with the scaling function

$$\mathcal{A}(\lambda_w^+; \lambda^+; \alpha) = e^{-\lambda_w^+} \left[ \frac{1}{\alpha} \delta(\lambda^+) + \left( 1 - \frac{1}{\alpha} \right) \theta(\lambda^+ > 0) e^{-\lambda^+} \right]. \quad (118)$$

Since we have the same properties for the half line  $x < 0$ , the measure for the full line reads

$$\begin{aligned} \mathcal{A}(\lambda_w^+; \lambda_w^-; \lambda^+; \lambda^-; \alpha) &= \theta(\lambda_w^+ > 0) \theta(\lambda_w^- > 0) e^{-\lambda_w^+ - \lambda_w^-} \\ &\times \left[ \frac{1}{\alpha} \delta(\lambda^+) + \left( 1 - \frac{1}{\alpha} \right) \theta(\lambda^+ > 0) e^{-\lambda^+} \right] \\ &\times \left[ \frac{1}{\alpha} \delta(\lambda^-) + \left( 1 - \frac{1}{\alpha} \right) \theta(\lambda^- > 0) e^{-\lambda^-} \right]. \end{aligned} \quad (119)$$

#### D. Rules for the two-time effective dynamics

Given the configuration  $(\lambda^-, \lambda_w^-, \lambda_w^+, \lambda^+)$  characterizing the renormalized landscape at the two successive scales, the two-time diffusion front takes the scaling form

$$\begin{aligned} P(x, t; x_w, t_w | 0, 0) &= \frac{1}{R_w^\mu R^\mu} \mathcal{P}_{\{\lambda^-, \lambda_w^-, \lambda_w^+, \lambda^+\}}^{(\mu)} \\ &\times \left( X_w = \frac{x_w}{R_w^\mu}, Y = \frac{x - x_w}{R^\mu}, F = \beta f R^\mu, \alpha = \frac{R^\mu}{R_w^\mu} \right) \end{aligned} \quad (120)$$

with

$$\begin{aligned} \mathcal{P}_{\{\lambda^-, \lambda_w^-, \lambda_w^+, \lambda^+\}}^{(0)}(X_w, Y, F, \alpha) &= \frac{\lambda_w^-}{\lambda_w^+ + \lambda_w^-} \delta(X_w - \lambda_w^+) \left[ W_+ \left( \lambda^+, \lambda^- + \frac{\lambda_w^+ + \lambda_w^-}{\alpha}, F \right) \right. \\ &\times \delta(Y - \lambda^+) + W_- \left( \lambda^+, \lambda^- + \frac{\lambda_w^+ + \lambda_w^-}{\alpha}, F \right) \\ &\times \delta \left( Y + \lambda^- + \frac{\lambda_w^+ + \lambda_w^-}{\alpha} \right) \left. \right] + \frac{\lambda_w^+}{\lambda_w^+ + \lambda_w^-} \delta(X_w + \lambda_w^-) \\ &\times \left[ W_+ \left( \lambda^+ + \frac{\lambda_w^+ + \lambda_w^-}{\alpha}, \lambda^-, F \right) \right. \\ &\times \delta \left( Y - \lambda^+ - \frac{\lambda_w^+ + \lambda_w^-}{\alpha} \right) \\ &\left. + W_- \left( \lambda^+ + \frac{\lambda_w^+ + \lambda_w^-}{\alpha}, \lambda^-, F \right) \delta(Y + \lambda^-) \right]. \end{aligned} \quad (121)$$

We are now in position to compute various observables in the aging regime from the knowledge of the sample-dependent two-time diffusion front (122) and from the measure (120) over the samples.

### VII. OBSERVABLES WHEN THE FIELD $f$ IS APPLIED FOR $t \geq t_w$

#### A. Disorder averaged two-time diffusion front

The disorder averaged diffusion front with the measure (120) reads

$$\begin{aligned} \overline{\mathcal{P}^{(0)}(X_w, Y, F, \alpha)} &\equiv \int_0^{+\infty} d\lambda_w^+ \int_0^{+\infty} d\lambda_w^- \int_0^{+\infty} d\lambda^+ \\ &\times \int_0^{+\infty} d\lambda^- \mathcal{A}(\lambda_w^+; \lambda_w^-; \lambda^+; \lambda^-; \alpha) \\ &\times \mathcal{P}_{\{\lambda^-, \lambda_w^-, \lambda_w^+, \lambda^+\}}^{(0)}(X_w, Y, F, \alpha) \\ &= [\theta(X_w \geq 0) \theta(Y \geq 0) + e^{-F|Y|} \theta(X_w \leq 0) \theta(Y \leq 0)] \\ &\times g_{++}^{(0)}(|X_w|, |Y|, F, \alpha) + [\theta(X_w \leq 0) \theta(Y \geq 0) \\ &+ e^{-F|Y|} \theta(X_w \geq 0) \theta(Y \leq 0)] g_{-+}^{(0)}(|X_w|, |Y|, F, \alpha), \end{aligned} \quad (122)$$

where

$$\begin{aligned} g_{++}^{(0)}(X_w, Y, F, \alpha) &= e^{-X_w} \left[ \frac{1}{\alpha} \delta(Y) + \left( 1 - \frac{1}{\alpha} \right) e^{-Y} \right] \int_0^{+\infty} d\lambda_w^- e^{-\lambda_w^-} \\ &\times \frac{\lambda_w^-}{X_w + \lambda_w^-} \left[ \left( \frac{1}{\alpha} \right) \frac{1 - e^{-F(X_w + \lambda_w^-/\alpha)}}{1 - e^{-F(Y + X_w + \lambda_w^-/\alpha)}} \right. \\ &\left. + \left( 1 - \frac{1}{\alpha} \right) \int_0^{+\infty} d\lambda^- e^{-\lambda^-} \frac{1 - e^{-F(\lambda^- + X_w + \lambda_w^-/\alpha)}}{1 - e^{-F(Y + \lambda^- + X_w + \lambda_w^-/\alpha)}} \right] \end{aligned} \quad (123)$$

and

$$\begin{aligned} g_{-+}^{(0)}(X_w, Y, F, \alpha) &= e^{-X_w} \left( 1 - \frac{1}{\alpha} \right) \int_0^{+\infty} d\lambda^- e^{-\lambda^-} \frac{1 - e^{-F\lambda^-}}{1 - e^{-F(Y + \lambda^-)}} \\ &\times \int_0^{+\infty} d\lambda_w^+ e^{-\lambda_w^+} \frac{\lambda_w^+}{\lambda_w^+ + X_w} \left[ \frac{1}{\alpha} \delta \left( Y - \frac{\lambda_w^+ + X_w}{\alpha} \right) \right. \\ &\left. + \left( 1 - \frac{1}{\alpha} \right) \theta \left( Y - \frac{\lambda_w^+ + X_w}{\alpha} \right) e^{-(Y - \lambda_w^+ + X_w/\alpha)} \right]. \end{aligned} \quad (124)$$

The form (122) presents the simple property

$$\overline{\mathcal{P}^{(0)}(-X_w, -Y, F, \alpha)} = e^{-FY} \overline{\mathcal{P}^{(0)}(X_w, Y, F, \alpha)} \quad (125)$$

which is expected to be true for arbitrary  $\mu$ , as a consequence of the nonlinear fluctuation theorem discussed in Ref. [23].

### B. Law of the relative displacement $Y$ between $t_w$ and $t$

We now consider the partial law of the rescaled relative displacement  $Y$  between  $t_w$  and  $t$ , i.e., we integrate the disorder averaged diffusion front (122) over the position  $X_w$  at time  $t_w$ : we obtain the form

$$\begin{aligned} \overline{\mathcal{P}^{(0)}(Y, F, \alpha)} &\equiv \int_{-\infty}^{+\infty} dX_w \overline{\mathcal{P}^{(0)}(X_w, Y, F, \alpha)} = [\theta(Y \geq 0) \\ &+ e^{-F|Y|} \theta(Y \leq 0)] G(Y, F, \alpha), \end{aligned} \quad (126)$$

where

$$\begin{aligned} G(Y, F, \alpha) &= \int_0^{+\infty} dX_w [g_{++}^{(0)}(|X_w|, |Y|, F, \alpha) \\ &+ g_{-+}^{(0)}(|X_w|, |Y|, F, \alpha)]. \end{aligned} \quad (127)$$

Using the intermediate results

$$\begin{aligned} &\int_0^{+\infty} dX_w g_{++}^{(0)}(X_w, Y, F, \alpha) \\ &= \frac{1}{2\alpha} \delta(Y) + \frac{1}{2} e^{-Y} \int_0^{+\infty} du \frac{1 - e^{-Fu}}{1 - e^{-F(Y+u)}} \\ &\quad \times [e^{-u} - e^{-\alpha u}] \end{aligned} \quad (128)$$

and

$$\begin{aligned} \int_0^{+\infty} dX_w g_{-+}^{(0)}(|X_w|, |Y|, F, \alpha) &= \frac{1}{2} (e^{-Y} - e^{-\alpha Y}) \int_0^{+\infty} du e^{-u} \\ &\quad \times \frac{1 - e^{-Fu}}{1 - e^{-F(Y+u)}}, \end{aligned} \quad (129)$$

we finally obtain

$$\begin{aligned} \overline{\mathcal{P}^{(0)}(Y, F, \alpha)} &= \frac{1}{\alpha} \delta(Y) + [\theta(Y \geq 0) \\ &+ e^{-F|Y|} \theta(Y \leq 0)] G_{ns}(|Y|, F, \alpha), \end{aligned} \quad (130)$$

where the nonsingular part is given in terms of the function

$$\begin{aligned} G_{ns}(Y, F, \alpha) &= \frac{e^{-Y}}{2} \int_0^{+\infty} du \frac{1 - e^{-Fu}}{1 - e^{-F(Y+u)}} \\ &\quad \times [(2 - e^{-(\alpha-1)Y}) e^{-u} - e^{-\alpha u}] \\ &= e^{-Y} \sum_{n=0}^{+\infty} e^{-FnY} \left[ \frac{H}{(1+Fn)(1+F+Fn)} \right. \end{aligned}$$

$$\left. \times (2 - e^{-(\alpha-1)Y}) - \frac{H}{(a+Fn)(a+F+Fn)} \right]. \quad (131)$$

The presence of a singular part in  $\delta(Y)$  for the rescaled relative displacement was already found in the case of the Sinai model by the RSRG approach [26] and by mathematicians [35]. Here the weight of  $\delta(Y)$  is simply given by  $(1/\alpha)$  (112) which represents the probability for a renormalized trap at scale  $\xi(f=0, t_w)$  to be still present in the renormalized landscape at the new scale  $\xi(f, t-t_w)$ .

### C. Disorder averaged mean position

Since  $\langle x_w \rangle = 0$  in any sample, in the effective dynamics in the limit  $\mu \rightarrow 0$  but also more generally for any  $\mu$  as a consequence of a special dynamical symmetry [23], the mean position  $\langle x \rangle$  at time  $t$  in a sample also corresponds to the mean displacement  $\langle (x - x_w) \rangle$  between the times  $t_w$  and  $t$ . The disorder average of the mean position takes the scaling form the scaling form

$$\overline{\langle x \rangle}(t, t_w, f) = R^\mu \mathcal{X}_\mu \left( F = \beta f R^\mu, \alpha = \frac{R^\mu}{R_w^\mu} \right), \quad (132)$$

where the scaling function is simply given in terms of the law of the relative displacement  $Y$  (130)

$$\begin{aligned} \mathcal{X}_0(F, \alpha) &= \frac{1}{2} \int_0^{+\infty} dv \frac{1}{1 - e^{-Fv}} \int_0^v dY Y e^{-Y} (1 - e^{-FY}) \\ &\quad \times (1 - e^{FY-Fv}) [(2 - e^{-(\alpha-1)Y}) e^{Y-v} - e^{\alpha Y - \alpha v}] \\ &= 1 - \frac{1}{F} - \frac{1}{F^3} \psi'' \left( 1 + \frac{1}{F} \right) - \frac{1}{2\alpha} - \frac{1}{2\alpha^2} \\ &\quad + \frac{1}{2(\alpha-1+F)} - \frac{1}{2\alpha^2(\alpha-1-F)} \\ &\quad + \frac{1}{(\alpha-1)((\alpha-1)^2 - F^2)} \left[ \psi' \left( 1 + \frac{1}{F} \right) \right. \\ &\quad \left. - \psi' \left( 1 + \frac{\alpha}{F} \right) \right]. \end{aligned} \quad (133)$$

We now consider various limit expressions.

On one hand, for fixed  $\alpha$ , the asymptotic expressions for small  $F$  and large  $F$  which generalize (82) and (83) read

$$\begin{aligned} \mathcal{X}_0(F, \alpha) &= \left[ 1 - \frac{1}{\alpha^2} \right] \frac{F}{2} - \frac{1}{6} \left[ 1 - \frac{1}{\alpha^3} \right] F^3 \\ &\quad + \frac{1}{6} \left[ 1 - \frac{1}{5} \left( \frac{1}{\alpha^3} + \frac{3}{\alpha^4} + \frac{1}{\alpha^5} \right) \right] F^5 + O(F^7) \end{aligned} \quad (134)$$

and

$$\begin{aligned} \mathcal{X}_0(F, \alpha) = & \left(1 - \frac{1}{2\alpha} - \frac{1}{2\alpha^2}\right) - \frac{1}{2} \left(1 - \frac{1}{\alpha^2}\right) \frac{1}{F} \\ & - \frac{1}{2} \left(\alpha - 1 - \frac{1}{\alpha} + \frac{1}{\alpha^2}\right) \frac{1}{F^2} + O\left(\frac{1}{F^3}\right). \end{aligned} \quad (135)$$

On the other hand, for fixed  $F$ , we have at the beginning of the aging regime where  $(\alpha - 1)$  is small

$$\begin{aligned} \mathcal{X}_0(F, \alpha) = & (\alpha - 1) \left[ \frac{3}{2} - \frac{1}{F} + \frac{1}{2F^4} \psi''' \left(1 + \frac{1}{F}\right) \right] \\ & + O((\alpha - 1)^2), \end{aligned} \quad (136)$$

whereas for large  $\alpha$ , the first corrections with respect to the previous result (81) corresponding to  $\alpha \rightarrow \infty$  read

$$\begin{aligned} \mathcal{X}_0(F, \alpha) = & 1 - \frac{1}{F} - \frac{1}{F^3} \psi'' \left(1 + \frac{1}{F}\right) - \frac{F}{2\alpha^2} \\ & + \left[ -F + \frac{F^2}{2} + \psi' \left(1 + \frac{1}{F}\right) \right] \frac{1}{\alpha^3} + O\left(\frac{1}{\alpha^4}\right) \end{aligned} \quad (137)$$

#### D. Mean-square displacement during $[t_w, t]$

Similarly, the mean-square displacement during  $[t_w, t]$  takes the scaling form

$$D(t, t_w) = \overline{[x(t) - x(t_w)]^2} = R^{2\mu} \mathcal{D}_\mu \left( F = \beta f R^\mu, \alpha = \frac{R^\mu}{R_w^\mu} \right). \quad (138)$$

where the scaling function is again obtained from the law of the relative displacement  $Y$  (130)

$$\begin{aligned} \mathcal{D}_0(F, \alpha) = & \int_0^{+\infty} dY Y^2 [1 + e^{-FY}] \frac{e^{-Y}}{2} \int_0^{+\infty} du \frac{1 - e^{-Fu}}{1 - e^{-F(Y+u)}} \\ & \times [(2 - e^{-(\alpha-1)Y}) e^{-u} - e^{-\alpha u}] \\ = & 2 - \frac{2}{F} + \frac{2}{F^2} - \frac{1}{\alpha} - \frac{1}{\alpha^3} + \frac{1}{\alpha - 1 + F} \\ & - \frac{1}{(\alpha - 1 + F)^2} - \frac{1}{\alpha^2(\alpha - 1 - F)^2} \\ & - \frac{1}{\alpha^3(\alpha - 1 - F)} + \frac{4(\alpha - 1)}{F((\alpha - 1)^2 - F^2)} \\ & \times \left[ \psi' \left(1 + \frac{1}{F}\right) - \psi' \left(1 + \frac{\alpha}{F}\right) \right] \\ & + \frac{2(\alpha - 1)^2 - F^2}{F^4((\alpha - 1)^2 - F^2)} \psi'' \left(1 + \frac{1}{F}\right) \end{aligned}$$

$$+ \frac{1}{F^2((\alpha - 1)^2 - F^2)} \psi'' \left(1 + \frac{\alpha}{F}\right). \quad (139)$$

For fixed  $\alpha$ , the asymptotic expressions for small  $F$ , and large  $F$  respectively, read

$$\begin{aligned} \mathcal{D}_0(F, \alpha) = & 1 - \frac{1}{\alpha^2} + \left[ \frac{1}{3} - \frac{1}{2\alpha^2} + \frac{2}{3\alpha^3} - \frac{1}{2\alpha^4} \right] F^2 \\ & + \left[ -\frac{1}{3} + \frac{1}{6\alpha^2} + \frac{1}{5\alpha^3} - \frac{2}{5\alpha^4} + \frac{1}{5\alpha^5} + \frac{1}{6\alpha^6} \right] \\ & \times F^4 + O(F^6) \end{aligned} \quad (140)$$

and

$$\begin{aligned} \mathcal{D}_0(F, \alpha) = & 2 - \frac{1}{\alpha} - \frac{1}{\alpha^3} - \left(1 - \frac{1}{\alpha^3}\right) \frac{1}{F} + \left(2 - \alpha - \frac{1}{\alpha^3}\right) \frac{1}{F^2} \\ & + O\left(\frac{1}{F^3}\right). \end{aligned} \quad (141)$$

For fixed  $F$ , we have at the beginning of the aging regime, where  $(\alpha - 1)$

$$\begin{aligned} \mathcal{D}_0(F, \alpha) = & \left[ 4 - \frac{3}{F} + \frac{2}{F^2} - \frac{1}{F^5} \psi''' \left(1 + \frac{1}{F}\right) \right] (\alpha - 1) \\ & + O((\alpha - 1)^2), \end{aligned} \quad (142)$$

whereas for large  $\alpha$ , we obtain the first corrections to the previous result (93)

$$\begin{aligned} \mathcal{D}_0(F, \alpha) = & \left[ 2 - \frac{2}{F} + \frac{2}{F^2} + \frac{2}{F^4} \psi'' \left(1 + \frac{1}{F}\right) \right] \\ & + \left[ -F + \frac{1}{F^2} \psi'' \left(1 + \frac{1}{F}\right) \right] \frac{1}{\alpha^2} \\ & + \left[ -2 + F^2 + \frac{4}{F} \psi' \left(1 + \frac{1}{F}\right) \right. \\ & \left. + \frac{2}{F^2} \psi'' \left(1 + \frac{1}{F}\right) \right] \frac{1}{\alpha^3} + O\left(\frac{1}{\alpha^4}\right). \end{aligned} \quad (143)$$

#### E. Thermal width

The rescaled thermal width

$$\Delta_\mu(F, \alpha) = \overline{\langle X^2 \rangle} - \langle X \rangle^2 \quad (144)$$

may be similarly computed, but since the full expression is rather lengthy, we will only give the asymptotic forms.

For fixed  $\alpha$ , the asymptotic expressions for small  $F$  and large  $F$ , respectively, read

$$\begin{aligned} \Delta_0(F, \alpha) = & 1 + \left[ -\frac{2}{3} + \frac{1}{\alpha^3} - \frac{1}{3\alpha^4} \right] F^2 \\ & + \left[ 1 - \frac{5}{3\alpha^4} + \frac{4}{9\alpha^5} + \frac{2}{9\alpha^6} \right] F^4 + O(F^6) \end{aligned} \quad (145)$$

and

$$\begin{aligned} \Delta_0(F, \alpha) = & \frac{1}{3\alpha} + \frac{1}{3\alpha^2} + \frac{1}{3\alpha^3} + \left( \frac{2}{3} - \frac{2}{3\alpha^3} \right) \frac{1}{F} \\ & + \left( \frac{\alpha}{6} - \frac{1}{6} - \frac{5}{6\alpha^2} + \frac{5}{6\alpha^3} \right) \frac{1}{F^2} + O\left(\frac{1}{F^3}\right). \end{aligned} \quad (146)$$

For fixed  $F$ , at the beginning of the aging regime where  $(\alpha - 1)$  is small, we have

$$\begin{aligned} \Delta_0(F, \alpha) = & 1 + \left[ -2 + \frac{2}{F} - \frac{2}{3F^2} - \frac{2}{3F^5} \psi''' \left( 1 + \frac{1}{F} \right) \right. \\ & \left. - \frac{1}{3F^6} \psi'''' \left( 1 + \frac{1}{F} \right) \right] (\alpha - 1) + O((\alpha - 1)^2), \end{aligned} \quad (147)$$

whereas for large  $\alpha$ , we obtain the first corrections to the previous result (87)

$$\begin{aligned} \Delta_0(F, \alpha) = & \frac{1}{F} + \frac{2}{F^4} \psi'' \left( 1 + \frac{1}{F} \right) + \frac{1}{F^5} \psi''' \left( 1 + \frac{1}{F} \right) + \frac{F^2}{\alpha^3} \\ & + \left[ 3F^2 - \frac{10}{3}F^3 + \frac{10}{3} \psi'' \left( 1 + \frac{1}{F} \right) \right] \frac{1}{\alpha^4} + O\left(\frac{1}{\alpha^5}\right). \end{aligned} \quad (148)$$

For the case  $t_w = 0$ , corresponding to  $\alpha = \infty$  we have previously found a very simple relation (88) between the two scaling functions describing the mean position and the thermal width. Here, in the aging case  $t_w > 0$  with finite  $\alpha$ , there does not seem to exist a simple generalization of Eq. (88).

### VIII. VARIOUS REGIMES FOR THE RESPONSE IN THE LIMIT $\mu \rightarrow 0$

In this section, we translate our results for the disorder averaged mean position [(132) and (133)] into the original unrescaled quantities, using the definitions for  $F = \beta f \xi(t, f)$  and for  $\alpha(t, t_w, f)$  (112). We have to distinguish various regimes, according to the relative values of the times  $(t, t_w)$  and of the characteristic time scale  $t_\mu(f)$  (55) associated to the force  $f$ .

#### A. Case $t_w \ll t_\mu(f)$

For  $t_w \ll t_\mu(f)$ , we obtain from Eqs. (134) and (113) the behavior in the sector  $t_w < t - t_w \ll t_\mu(f)$ :

$$\overline{\langle x \rangle}(t, t_w, f) \simeq \frac{\beta f}{2} (t - t_w)^{\mu/(1+\mu)} \left[ 1 - \left( \frac{t_w}{t - t_w} \right)^{2\mu/(1+\mu)} \right] \quad (149)$$

and from Eqs. (135) and (114) the behavior at long times  $t - t_w \gg t_\mu(f)$ :

$$\begin{aligned} \overline{\langle x \rangle}(t, t_w, f) \simeq & \left( \frac{\beta f}{2} \right)^\mu (t - t_w)^\mu \left[ 1 - \frac{1}{2} \left( \frac{t_w^{1/(1+\mu)}}{\beta f/2(t - t_w)} \right)^\mu \right. \\ & \left. - \frac{1}{2} \left( \frac{t_w^{1/(1+\mu)}}{\beta f/2(t - t_w)} \right)^{2\mu} \right]. \end{aligned} \quad (150)$$

Moreover, at the beginning of the effective dynamics regime  $\alpha \rightarrow 1$  (113), the expressions (134) and (136) coincide and give

$$\overline{\langle x \rangle}(t, t_w, f) \underset{t-t_w \rightarrow t_w}{\simeq} \beta f t_w^{\mu/(1+\mu)} \left[ \left( \frac{t - t_w}{t_w} \right)^{\mu/(1+\mu)} - 1 \right], \quad (151)$$

whereas asymptotically when  $\alpha \rightarrow \infty$  (114), the expressions (135) and (137) give

$$\overline{\langle x(t, f) \rangle} \underset{t-t_w \gg \frac{2}{\beta f} t_w^{1/(1+\mu)}}{\simeq} \left( \frac{\beta f}{2} \right)^\mu (t - t_w)^\mu \quad (152)$$

#### B. Case $t_w \gg t_\mu(f)$

For  $t_w \gg t_\mu(f)$ , the time sector  $\alpha > 1$  (114) implies that  $t - t_w \gg t_\mu(f)$  and we obtain from Eqs. (135) and (114) the behavior at long times in time sector  $t - t_w \gg t_\mu(f)$  and  $t - t_w > \frac{2}{\beta f} t_w^{1/(1+\mu)}$

$$\begin{aligned} \overline{\langle x \rangle}(t, t_w, f) = & \left( \frac{\beta f}{2} \right)^\mu (t - t_w)^\mu \left[ 1 - \frac{1}{2} \left( \frac{t_w^{1/(1+\mu)}}{\beta f/2(t - t_w)} \right)^\mu \right. \\ & \left. - \frac{1}{2} \left( \frac{t_w^{1/(1+\mu)}}{\beta f/2(t - t_w)} \right)^{2\mu} \right]. \end{aligned} \quad (153)$$

Here at the beginning of the aging regime  $t - t_w \rightarrow (2/\beta f) t_w^{1/(1+\mu)}$ , the expressions (135) and (136) coincide and give

$$\overline{\langle x \rangle}(t, t_w, f) \underset{t-t_w \rightarrow \frac{2}{\beta f} t_w^{1/(1+\mu)}}{\simeq} \frac{3}{2} t_w^{\mu/(1+\mu)} \left[ \left( \frac{\beta f}{2} (t - t_w) \right)^\mu \frac{1}{t_w^{1/(1+\mu)}} - 1 \right] \quad (154)$$

whereas at the end of the aging regime  $\alpha \rightarrow \infty$  (114), the expressions (135) and (137) again give expression (152).

### C. Discussion

The results of this section, that rely on the effective dynamics picture, are valid at lowest order in the high disorder limit  $\mu \rightarrow 0$ , and in the asymptotic aging regime where the two times are big  $t \rightarrow \infty$ ,  $t_w \rightarrow \infty$ , with the parameter  $\alpha(t, t_w, f) > 1$  (112) being fixed. We have found various interesting behaviors depending on the relative values of the parameters  $(t, t_w, f)$ . Whenever they can be compared, our results agree with the scaling analysis presented in Ref. [22]. In the following section, we discuss the behavior of the response in the time sector  $\alpha < 1$ , which is governed by rare events.

## IX. RESPONSE IN THE TIME SECTOR $\alpha(t, t_w, f) < 1$ FROM RARE EVENTS

As explained before, in the time sector  $\alpha(t, t_w, f) < 1$  (112), the effective dynamics governed by the decimation procedure gives no contribution, and the response will thus be governed by rare events. A similar situation was already found in the RSRG studies on the Sinai model [26] and on the out-of-equilibrium dynamics of the random field Ising model [28].

### A. Description of the rare events

For the trap model considered here, the ‘‘rare events’’ that are responsible for the response in the time sector  $\alpha(t, t_w, f) < 1$ , can be described as follows: the particle, which is assumed to be trapped in a renormalized trap  $\tau > R(t_w, 0)$  at time  $t_w$  in the effective dynamics, has actually a small probability to be at  $t_w$  in a ‘‘small’’ trap, i.e., an already decimated trap  $\tau < R(t_w, 0)$ , for two reasons.

(i) When  $\mu$  is small but finite, the particle can be found at time  $t_w$  in a trap  $\tau < R_w$  with a probability of order  $\mu$ , as explained in Ref. [9], where the corrections in  $\mu$  with respect to the effective dynamics of the unbiased trap model were studied in details.

(ii) When  $t_w$  is large but not infinite, there is a small probability that the particle is doing an excursion [(34) and (38)] at time  $t_w$ .

In both cases, the particle that happens to be in a small trap at  $t_w$  will respond to the external field in the time sector  $\alpha < 1$ .

### B. Correction of order $\mu$ to the effective dynamics

In the previous study on the unbiased trap model [9], we have studied in details the first corrections at order  $\mu$  to the effective dynamics. In particular, we have shown that in the asymptotic time regime, the particle can be found with a probability of order  $\mu$  in the biggest trap  $S$  contained in the interval  $]M_-, M_+[$  between the two renormalized traps around the origin at scale  $R(t_w, f = 0)$ . In particular, we have

obtained that the probability  $\psi_{R_w}^{(\mu)}(\tau)$  to be at  $t_w$  in a trap of trapping time  $\tau$  was given in the domain  $\tau < R_w$  at first order in  $\mu$  by

$$\psi_{R_w}^{(\mu)}(\tau) \theta(\tau < R_w) \simeq \mu \frac{R_w^{3/2}}{\tau^{5/2}} K_1 \left( \sqrt{\frac{R_w}{\tau}} \right) K_2 \left( \sqrt{\frac{R_w}{\tau}} \right), \quad (155)$$

where we have dropped numerical factors of order 1 (we refer the reader to Ref. [9] for precise results). In particular, this probability presents an essential singularity at the origin

$$\psi_{R_w}^{(\mu)}(\tau) \simeq_{\tau \rightarrow 0} \mu \frac{R_w}{\tau^2} e^{-2\sqrt{R_w/\tau}}, \quad (156)$$

which means that it is very unlikely for the particle to be trapped at  $t_w$  in the biggest trap  $S$  contained in the interval  $]M_-, M_+[$  if the trapping time  $\tau$  of  $S$  is much smaller than  $R_w$ .

As a consequence, the integrated probability up to scale  $R$  (with  $R < R_w$ ):

$$\int^R d\tau \psi_{R_w}^{(\mu)}(\tau) \sim \mu \int_{1/\sqrt{\alpha}}^{+\infty} dz z^2 K_1(z) K_2(z) \quad (157)$$

that is of order  $\mu$  when  $\alpha \sim 1$ , will also present an essential singularity for small  $\alpha = R/R_w$ :

$$\int^R d\tau \psi_{R_w}^{(\mu)}(\tau) \sim_{\alpha \rightarrow 0} \mu \frac{1}{\sqrt{\alpha}} e^{-2/\sqrt{\alpha}} \quad (158)$$

### C. Probability to be doing an excursion at $t_w$

#### 1. Probability to be doing an unsuccessful excursion at $t_w$

From the discussion on unsuccessful excursions, the probability for a particle to be trapped in the vicinity of a renormalized trap  $\tau_0$  at scale  $R$ , with neighbors at distances  $l_{\pm}$  reads when the external field vanishes (34)

$$P_R^{out}(\tau_0, l_+, l_-, \lambda_-) \simeq \frac{t_{out}}{t_{in}} \simeq \frac{l_+ + l_-}{6\tau_0}. \quad (159)$$

After the average over  $\tau_0$  with the measure (14) and over the lengths (13), we obtain that the probability to be doing an unsuccessful excursion at time  $t_w$  reads (110)

$$P^{out}(t_w) \equiv \overline{P_R^{out}(\tau_0, l_+, l_-, \lambda_-)} \sim \mu R_w^{\mu-1} = \mu t_w^{(\mu-1)/(1+\mu)}. \quad (160)$$

As expected, the probability of these rare events is very small when  $t_w$  is large since  $\mu < 1$ . In addition, there is a prefactor  $\mu$  that makes this probability even smaller in the limit  $\mu \rightarrow 0$ .

#### 2. Probability to be doing a successful excursion at $t_w$

The particle may be in a successful excursion at  $t_w$ , if it belongs to a renormalized trap that gets decimated around the scale  $R_w$ . From the discussion on successful excursions,

the diffusion time for a length  $l \sim R_w^\mu$  reads when the external field vanishes reads [(A22) and (A24)]

$$t_{diff}(f=0, l) \simeq \frac{l^2}{6} \sim R_w^{2\mu}. \quad (161)$$

This time window of width  $(\Delta t)_w \sim R_w^{2\mu} \sim t_w^{2\mu/(1+\mu)}$  around  $t_w$  corresponds in RG scale to a window around  $R_w \sim t_w^{1/(1+\mu)}$  of width  $\Delta R_w \sim t_w^{-\mu/(1+\mu)} (\Delta t)_w \sim t_w^{\mu/(1+\mu)}$ . As a consequence, the probability to be doing a successful excursion at  $t_w$  can be obtained from the probability to be in a trap  $\tau \sim R_w$  (14) times the window width just estimated

$$P^{diff}(t_w) \sim q_{R_w}(R_w) \Delta R_w \sim t_w^{(\mu-1)/(1+\mu)}. \quad (162)$$

The probability is again very small as expected, and happens to have exactly the same scale as (160).

### 3. Probability to be in a small trap $\tau$ during an excursion

For a particle doing an excursion, we are now interested into the probability  $\psi_{R_w}^{exc}(\tau)$  to be in a trap of trapping time  $\tau$ . We have obtained above that the total probability to be in a small trap  $\tau < R_w$  behaves as

$$\int^{R_w} d\tau \psi_{R_w}^{exc}(\tau) \sim \mu t_w^{(\mu-1)/(1+\mu)} \sim \mu R_w^{\mu-1}. \quad (163)$$

Assuming that the dependence of  $\psi_{R_w}^{exc}(\tau)$  in  $\tau$  is given by  $\tau q(\tau)$ , i.e., the initial distribution (6) weighed by the trapping time  $\tau$  (i.e., the particle spends in each trap a time proportional to its trapping time), we obtain the estimation

$$\psi_{R_w}^{exc}(\tau) \sim R_w^{2(\mu-1)} \frac{\mu}{\tau^\mu} \theta(\tau < R_w), \quad (164)$$

which is rather flat for small  $\mu \rightarrow 0$ , in contrast with Eq. (156). In particular, the integrated probability up to scale  $R$  (with  $R < R_w$ ) reads

$$\int^R d\tau \psi_{R_w}^{exc}(\tau) \sim R_w^{2(\mu-1)} R^{1-\mu} = R_w^{(\mu-1)} \alpha^{1-\mu}. \quad (165)$$

### D. Contribution to the response of these rare events

When the initial condition at  $t_w$  is a small trap of trapping time  $\tau$ , the effective dynamics will become active again when  $R(f, t - t_w)$  reaches  $\tau$ , i.e., when  $\alpha$  reaches  $\tau/R_w < 1$ , and the corresponding contribution to the response reads

$$\langle \overline{x} \rangle_{\tau}(t, t_w, f) = \theta(R > \tau) R \mathcal{X}_0(F = \beta f R^\mu), \quad (166)$$

where the condition  $\theta(R > \tau)$  means that the trap has been decimated at scale  $R$ , and thus the response is given in terms of the scaling function (81) found before for the case  $t_w = 0$ . Averaging over  $\tau$  with the total probability  $\psi_{R_w}(\tau) = \psi_{R_w}^{(\mu)}(\tau) + \psi_{R_w}^{exc}(\tau)$  coming from the two kinds of rare events described above, we obtain the leading term of the response in the sector  $\alpha = R/R_w < 1$  as

$$\begin{aligned} \langle \overline{x} \rangle_{rare}(t, t_w, f) &= \int^R d\tau \psi_{R_w}(\tau) \langle \overline{x} \rangle_{\tau}(t, t_w, f) \\ &= R \mathcal{X}_0(F = \beta f R^\mu) \left[ \int^R d\tau \psi_{R_w}^{(\mu)}(\tau) \right. \\ &\quad \left. + \int^R d\tau \psi_{R_w}^{exc}(\tau) \right]. \end{aligned} \quad (167)$$

Using the estimations of the integrated probabilities found before [(158) and (165)], we finally obtain that the contribution of excursions dominate for small  $\alpha < \alpha_w$ , whereas the contribution of the corrections in  $\mu$  to the effective dynamics are dominant for  $\alpha_w < \alpha < 1$ . The scale of the crossover value  $\alpha_w$  can be estimated from the equality between Eqs. (158) and (165) at leading order

$$\alpha_w \sim \frac{1}{(\ln R_w)^2}. \quad (168)$$

In conclusion, before the response of the effective dynamics in the sector  $\alpha > 1$  given in Eqs. (149), (150), and (153), there exists a response in the sector  $\alpha < 1$  as a consequence of rare events. For very small  $\alpha < \alpha_w$ , the response comes from the particles doing excursions in anomalously small traps at  $t_w$ , and it is reduced by a very small prefactor of order  $\mu t_w^{2(\mu-1)/(\mu+1)} (t - t_w)^{(1-\mu)/(\mu+1)}$  (165). On the other hand, for  $\alpha_w < \alpha < 1$ , the response is governed by particles which are “in delay” with respect to the effective dynamics, as a consequence of  $\mu$  being finite, and the response is reduced by only a factor of  $\mu$ .

## X. DISCUSSION OF THE FLUCTUATION-DISSIPATION RELATION

### A. Linear response regime in a given sample in the effective dynamics time sector

From the diffusion front (121) in a given sample characterized by  $(\lambda^+, \lambda_-, \lambda^+, \lambda_-)$ , we obtain at lowest order in  $F$  the following results for the rescaled mean position

$$\begin{aligned} \frac{\langle x \rangle(t, t_w, f)}{\xi(t - t_w, f)} &= \langle Y \rangle(\alpha, F) \\ &= \left[ \frac{\lambda^+ \lambda^-}{2} + \frac{\lambda^+ \lambda_w^- + \lambda_w^+ \lambda^-}{2\alpha} \right] F + O(F^2) \end{aligned} \quad (169)$$

and the rescaled mean-square displacement

$$\begin{aligned} \frac{\langle (x - x_w)^2 \rangle(t, t_w, f)}{\xi^2(t - t_w, f)} &= \langle Y^2 \rangle(\alpha, F) \\ &= \left[ \lambda^+ \lambda^- + \frac{\lambda^+ \lambda_w^- + \lambda_w^+ \lambda^-}{\alpha} \right] + O(F). \end{aligned} \quad (170)$$



So the Fluctuation-Dissipation Theorem (FDT) or Einstein's relation is valid in the whole time sector  $\alpha > 1$  as long as the linear response is valid and reads in unrescaled quantities with  $F = \beta f \xi(t - t_w, f)$ ,

$$\langle x \rangle(t, t_w, f) \underset{f \rightarrow 0}{\simeq} \frac{\beta f}{2} \langle (x - x_w)^2 \rangle(t, t_w, 0). \quad (171)$$

This is in agreement with the scaling arguments and numerical simulations presented in Ref. [22] and with the nonlinear fluctuation theorem discussed in Ref. [23], that proves that the FDT relation is valid in any given sample for arbitrary  $\mu$ .

The validity of the FDT relation for the trap model in its aging phase is nevertheless quite remarkable, since the dynamics is completely out-of-equilibrium: indeed, in the effective dynamics, the weights of the two important traps are not given by Boltzmann factors, they do not even depend on the energies of these two traps, but they are given by the probabilities to reach one before the other one, and they, thus, only depend on the distances to the origin. This example, with the explicit expressions in a given sample [(169) and (170)], thus shows that the validity of the FDT relation in the linear response regime does not imply that the system is at equilibrium or even near equilibrium.

### B. Nonlinear response in the asymptotic aging sector $\alpha \rightarrow \infty$

In the asymptotic aging sector  $\alpha \rightarrow \infty$ , which also corresponds to the case where the external field is applied from the very beginning  $t_w = 0$ , we have found a very simple relation (88) between the scaling functions for the mean displacement and the thermal width which is valid for arbitrary  $F$  and in particular in the whole nonlinear response regime. However, this relation found for disorder averaged quantities does not seem to have a simple interpretation, since in a given sample, there is not such a relation between the rescaled mean position and the rescaled thermal width that are given by Eq. (74)

$$\frac{\langle x(t, f) \rangle}{\xi(t, f)} = \frac{\lambda^+ (1 - e^{-F\lambda^-}) - \lambda^- e^{-F\lambda^-} (1 - e^{-F\lambda^+})}{1 - e^{-F(\lambda^+ + \lambda^-)}} \quad (172)$$

$$\frac{\langle \Delta x^2(t, f) \rangle}{\xi^2(t, f)} = (\lambda^+ + \lambda^-)^2 \frac{e^{-F\lambda^-} (1 - e^{-F\lambda^+}) (1 - e^{-F\lambda^-})}{(1 - e^{-F(\lambda^+ + \lambda^-)})^2}. \quad (173)$$

Moreover, we have not found an equivalent relation when  $\alpha$  is finite (144). Nevertheless, after the average over the samples, we have obtained the following simple property at very long times in the aging regime  $\alpha \rightarrow \infty$  (114), i.e., in the nonlinear response regime [(85) and (92)]

$$\lim_{t \rightarrow \infty} \left( \frac{\overline{\langle x \rangle(t, t_w, f)}}{\overline{\langle \Delta x^2(t, t_w, f) \rangle}} \right) = \beta f, \quad (174)$$

whereas, for comparison, there is a factor (1/2) in the FDT relation of the linear response regime (171)

$$\left( \frac{\overline{\langle x \rangle(t, t_w, f)}}{\overline{\langle \Delta x^2(t, t_w, f) \rangle}} \right)_{f \rightarrow 0} \simeq \frac{\beta f}{2}. \quad (175)$$

As a comparison, in the pure trap model, it is immediate to obtain the equations for the mean displacement ( $d\langle n \rangle_{pure}/dt = 2 \sinh(\beta f/2)$ ) and for the thermal width ( $d\langle \Delta n^2 \rangle_{pure}/dt = 2 \cosh(\beta f/2)$ ) so that their ratio is simply

$$\frac{\langle n \rangle_{pure}(t)}{\langle \Delta n^2 \rangle_{pure}(t)} = \tanh \frac{\beta f}{2} \simeq \frac{\beta f}{2} \quad (176)$$

for arbitrary time, in the regime of small asymmetry we are interested in Eq. (10).

## XI. CONCLUSION

We have studied in details the dynamics of the one-dimensional disordered trap model when an external force is applied from the very beginning at  $t=0$ , or only after a waiting time  $t_w$ , in the linear as well as in the nonlinear response regime. Using a disorder-dependent real-space renormalization procedure that becomes exact in the limit of strong disorder  $\mu \rightarrow 0$ , we have shown that the diffusion front in each sample consists in two- $\delta$  peaks, which are completely out of equilibrium with each other, since their weights represent the probabilities to reach one before the other one. The statistics of the positions and weights of these  $\delta$  peaks over the samples was then used to obtain explicit results for many observables, such as the diffusion front, the mean position, the thermal width, the localization parameters, and the two-particle correlation function. Since the renormalization procedure is defined sample by sample, our approach provides a very clear insight into the important dynamical processes.

From a more general perspective, it seems that up to now, the studies on the response of aging systems to an external field have been mainly restricted to the linear response regime [17,18], which holds for fixed times  $(t_w, t)$  in the limit of vanishing field  $f \rightarrow 0$ . However, as in the trap model discussed here, it should be expected in a broad class of aging systems that, for a fixed small field  $f$ , the validity of the linear response regime is limited in the time sector for  $(t_w, t)$  by a characteristic time  $\tau(f)$  depending on the external field. Indeed, it seems rather natural that an external field, even if it is arbitrarily small, will, for sufficiently long times, drive the system into a configurational landscape which is completely different from the initial one. So in the asymptotic time regime beyond the characteristic scale  $\tau(f)$ , the response will always be governed by nonlinear effects. For the special case  $t_w = 0$ , the full response including these nonlinear effects has already been studied for the Sinai model [26] as well as in the coarsening dynamics of the random field Ising model [28], via the RSRG approach: in both cases, as in the trap model, the field introduces a characteristic time separating the linear response regime from a nontrivial aging regime with nonlinear effects. This scenario should more generally apply to other coarsening dynamics. However, in numerical studies on domain growth processes, to get better results for

the linear response regime at large times, it is usually the response to a random field that is measured [36,37], and not the response to a constant field, that would favor one of the phase and induces nonlinear effects rapidly. Since this choice of a random field for domain growth processes, is in some sense the equivalent of a constant field for spin glasses [36], it seems that the nonlinear response of spin-glasses will be very different from coarsening systems. For instance, in the dynamics of the spherical Sherrington-Kirkpatrick (SK) spin-glass model [38], the magnetic field introduces a characteristic time that separates the aging dynamics of the linear response regime from an equilibrium dynamics at large times: here, the magnetic field does not lead at large times to a nontrivial aging regime with nonlinear effects in the field, but rather gives rise to an interrupted aging phenomenon. However, this scenario is not expected to hold for other spin glasses such as the usual SK model [38], if one considers the number of metastable states in a field [39]. In conclusion, the understanding of the nonlinear effects that arise at large times in the response of aging systems is still very incomplete and should give rise to further studies in the future.

#### ACKNOWLEDGMENTS

It is a pleasure to thank E. Bertin, J.P. Bouchaud, G. Biroli, and J.M. Luck for useful discussions.

#### APPENDIX: STATISTICAL PROPERTIES OF EXCURSIONS IN THE PRESENCE OF A FIELD

To study the excursions in the renormalized landscape in the presence of a field, we have to study the following standard problem: what is the probability distribution  $P_t(x)$  of the time  $t$  of the first passage at  $x=0$  without having touched the other boundary  $x=l$  before, for a pure random walk starting at  $x$  with the asymmetry  $h=h(f)>0$  (8)?

For  $x=1, \dots, l-1$ , the probability distribution  $P_t(x)$  satisfies the equation

$$\partial_t P_t(x) = (1+h)P_{x-1}(t) + (1-h)P_{x+1}(t) - 2P_t(x) \quad (\text{A1})$$

with the boundary conditions  $P_0(t) = \delta(t)$  and  $P_l(t) = 0$ . So the Laplace transform with respect to  $t$

$$\hat{P}_x(s) \equiv \int_0^{+\infty} dt e^{-st} P_t(x) \quad (\text{A2})$$

satisfies

$$(1+h)\hat{P}_{x+1}(s) + (1-h)\hat{P}_{x-1}(s) - (2+s)\hat{P}_x(s) = 0 \quad (\text{A3})$$

for  $x=1, \dots, l-1$  with the boundary conditions  $\hat{P}_0(s) = 1$  and  $\hat{P}_l(s) = 0$ .

The solution reads

$$\hat{P}_x(s) = \frac{\rho_+^l(s)\rho_-^x(s) - \rho_-^l(s)\rho_+^x(s)}{\rho_+^l(s) - \rho_-^l(s)} \quad (\text{A4})$$

in terms of the roots

$$\rho_{\pm}(s) = \frac{2+s \pm \sqrt{s^2+4s+4h^2}}{2(1+h)}. \quad (\text{A5})$$

The series expansion in  $s$  then yields the first moments

$$\theta_k(x) \equiv \int_0^{+\infty} dt t^k P_t(x). \quad (\text{A6})$$

#### 1. Escape probabilities

For  $s=0$ , the roots become (we assume  $h>0$ ) in terms of the bias  $f$  (8)

$$\rho_+(0) = 1 \quad (\text{A7})$$

$$\rho_-(0) = \frac{1-h}{1+h} = e^{-\beta f} \quad (\text{A8})$$

and the probability to reach 0 before  $l$  when starting at  $x$ , thus, reads

$$\theta_0(x) = \hat{P}_x(s=0) = \frac{e^{-\beta f x} - e^{-\beta f l}}{1 - e^{-\beta f l}}. \quad (\text{A9})$$

#### a. Escape probability along the drift

When starting at  $x=1$ , the probability to reach  $x=l$  without any visit to  $x=0$  reads

$$p_e^+(l, f) = 1 - \theta_0(x=1) = \frac{1 - e^{-\beta f}}{1 - e^{-\beta f l}}. \quad (\text{A10})$$

Here we are interested in the regime  $\beta f \ll 1$  and  $l \gg 1$ , where the escape probability takes the scaling form

$$p_e^+(l, f) \approx \frac{1}{l} E^+(u = \beta f l) \quad (\text{A11})$$

with the scaling function

$$E^+(u) = \frac{u}{1 - e^{-u}}. \quad (\text{A12})$$

In particular,  $E^+(u \rightarrow 0) \rightarrow 1$  corresponds to the unbiased case where the escape probability is simply  $1/l$ . In the other limit where  $u \rightarrow +\infty$ , we have  $E^+(u) \approx u$  and the escape probability becomes  $(\beta f)$ , i.e., it is proportional to the drift and independent of  $l$ .

#### b. Escape probability against the drift

When starting at  $x=l-1$ , the probability to escape to 0 without any visit to  $x=l$  reads

$$p_e^-(l, f) = \theta_0(x=l-1) = \frac{(1 - e^{-\beta f})e^{-\beta f(l-1)}}{1 - e^{-\beta fl}}, \quad (\text{A13})$$

which varies as it should between  $p_e^-(l, f \rightarrow 0) = 1/l$  for the unbiased case and  $p_e^-(l, f \rightarrow 1) = 0$  for the directed case. In the regime  $\beta f \ll 1$  and  $l \gg 1$ , the escape probability takes the scaling form

$$p_e^-(l, f) \simeq \frac{1}{l} E^-(u = \beta fl) \quad (\text{A14})$$

with the scaling function

$$E^-(u) = \frac{u}{e^u - 1}. \quad (\text{A15})$$

In particular,  $E^-(u \rightarrow 0) \rightarrow 1$  corresponds to the unbiased case where the escape probability is simply  $1/l$ . In the other limit where  $u \rightarrow +\infty$ , we have  $E^-(u) \simeq u e^{-u}$  and the escape probability becomes exponentially small  $p_e^-(l, f) \simeq \beta f e^{-\beta fl}$ .

## 2. Mean time for unsuccessful excursions

The expansion at first order in  $s$  of Eq. (A4) yields Eq. (A6) the mean time to reach 0 without any visit to  $x=l$  when starting at  $x$ :

$$\theta_1(x) = \frac{1 + e^{-\beta f}}{1 - e^{-\beta f}} \left[ \frac{x(e^{-\beta fx} + e^{-\beta fl})}{2(1 - e^{-\beta fl})} - \frac{le^{-\beta fl}(1 - e^{-\beta fx})}{(1 - e^{-\beta fl})^2} \right]. \quad (\text{A16})$$

### a. Unsuccessful excursion along the drift

For  $x=1$ , in the limit  $\beta f \ll 1$  and  $l \gg 1$ , the mean time of unsuccessful excursions along the drift takes the scaling form

$$\theta_{us}^+(f, l) = \theta_1(x=1) \simeq l \Theta(u = \beta fl), \quad (\text{A17})$$

where the scaling function

$$\Theta(u) = \frac{1 - 2ue^{-u} - e^{-2u}}{u(1 - e^{-u})^2} \quad (\text{A18})$$

interpolates between

$$\Theta(u) = \frac{1}{3} - \frac{u^2}{90} + O(u^4) \quad (\text{A19})$$

for the unbiased case, where the mean time is  $(l/3)$ , and

$$\Theta(u) \simeq_{u \rightarrow \infty} \frac{1}{u} \quad (\text{A20})$$

where the mean time is  $1/(\beta f)$ .

### b. Unsuccessful excursion against the drift

The unsuccessful excursions against the drift have the same properties

$$\theta_{us}^-(f, l) = \theta_{us}^+(f, l). \quad (\text{A21})$$

## 3. Mean time for the successful excursions

Similarly, we find that the mean time needed to reach  $x=l$  when starting at  $x=1$  for a random walk conditioned not to visit  $x=0$  takes the scaling form

$$t_{diff}^+(f, l) \simeq l^2 D(u = \beta fl), \quad (\text{A22})$$

where the scaling function

$$D(u) = \frac{u - 2 + (u+2)e^{-u}}{u^2(1 - e^{-u})} \quad (\text{A23})$$

interpolates between

$$D(u) = \frac{1}{6} + O(u^2) \quad (\text{A24})$$

for the unbiased case, where the mean diffusion time is  $(l^2/6)$ , and

$$D(u) = \frac{1}{u} + O\left(\frac{1}{u^2}\right), \quad (\text{A25})$$

where the mean diffusion time is  $l/(\beta f)$ .

Similarly, we find that the mean time for a successful excursion against the drift has the same properties

$$t_{diff}^-(f, l) = t_{diff}^+(f, l). \quad (\text{A26})$$

- [1] M.V. Feigelman and V. Vinokur, J. Phys. (France) **49**, 1731 (1988).  
 [2] J.P. Bouchaud, J. Phys. I **2**, 1705 (1992).  
 [3] J. P. Bouchaud, L. F. Cugliandolo, J. Kurchan, and M. Mézard, in *Spin Glasses and Random Fields*, edited by A. P. Young (World Scientific, Singapore, 1998).  
 [4] J.P. Bouchaud and D. Dean, J. Phys. I **5**, 265 (1995).  
 [5] C. Monthus and J.P. Bouchaud, J. Phys. A **29**, 3847 (1996).  
 [6] E. Barkai and Y.C. Cheng, e-print cond-mat/0211386.  
 [7] E. Bertin and J.P. Bouchaud, J. Phys. A **35**, 3039 (2002).

- [8] E.M. Bertin and J.P. Bouchaud, Phys. Rev. E **67**, 026128 (2003).  
 [9] C. Monthus, Phys. Rev. E **68**, 036114 (2003).  
 [10] G. Ben Arous, e-print cond-mat/0304364.  
 [11] L.R. Fontes, M. Isopi, and C.M. Newman, Ann. Prob. **30**, 579 (2002).  
 [12] G. Ben Arous and J. Cerny, e-print cond-mat/0210633.  
 [13] G. Ben Arous, J. Cerny, and T. Mountford (unpublished); J. Cerny, Ph.D. thesis (unpublished).  
 [14] S. Alexander, J. Bernasconi, W. Schneider, and R. Orbach,

- Rev. Mod. Phys. **53**, 175 (1981).
- [15] J.P. Bouchaud and A. Georges, Phys. Rep. **195**, 234 (1990).
- [16] T. Hwa, E. Marinari, K. Sneppen, and L.H. Tang, e-print cond-mat/0302603.
- [17] L.F. Cugliandolo, e-print cond-mat/0210312.
- [18] A. Crisanti and F. Ritort, e-print cond-mat/0212490.
- [19] S. Fielding and P. Sollich, Phys. Rev. Lett. **88**, 050603 (2002).
- [20] F. Ritort, e-print cond-mat/0303445.
- [21] P. Sollich, e-print cond-mat/0303637.
- [22] E.M. Bertin and J.P. Bouchaud, Phys. Rev. E **67**, 065105 (2003).
- [23] C. Monthus, e-print cond-mat/0305618.
- [24] C. Dasgupta and S.K. Ma Phys. Rev. B **22**, 1305 (1980).
- [25] D.S. Fisher, Phys. Rev. B **50**, 3799 (1994); D.S. Fisher, *ibid.* **51**, 6411 (1995).
- [26] D.S. Fisher, P. Le Doussal, and C. Monthus, Phys. Rev. Lett. **80**, 3539 (1998); P. Le Doussal, C. Monthus, and D.S. Fisher, Phys. Rev. E **59**, 4795 (1999).
- [27] P. Le Doussal and C. Monthus, Phys. Rev. E **60**, 1212 (1999).
- [28] D.S. Fisher, P. Le Doussal, and C. Monthus, Phys. Rev. E **64**, 066107 (2001).
- [29] C. Monthus, Phys. Rev. E **67**, 046109 (2003).
- [30] J. Kurchan, J. Phys. A **31**, 3719 (1998).
- [31] L.F. Cugliandolo, D.S. Dean, and J. Kurchan, Phys. Rev. Lett. **79**, 2168 (1997).
- [32] C. Monthus and P. Le Doussal, e-print cond-mat/0206035.
- [33] J.P. Bouchaud, A. Georges, and P. Le Doussal, J. Phys. (France) **48**, 1855 (1987).
- [34] J. Machta, J. Phys. A **18**, L531 (1985).
- [35] A. Dembo, A. Guionnet, and O. Zeitouni, e-print math-PR/0105215.
- [36] A. Barrat, Phys. Rev. E **57**, 3629 (1998).
- [37] F. Corberi, A. de Candia, E. Lippiello, and M. Zannetti, Phys. Rev. E **65**, 046114 (2002); F. Corberi, C. Castellano, E. Lippiello, and M. Zannetti, *ibid.* **65**, 066114 (2002).
- [38] L. Cugliandolo and D. Dean, J. Phys. A **28**, L453 (1995).
- [39] D. Dean, J. Phys. A **27**, L899 (1994).